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MECHANIZING PROOF THEORY: RESOURCE-AWARE LOGICS AND PROOF TRANSFORMATIONS TO EXTRACT IMPLICIT INFORMATION

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By
Gianluigi Bellin
June 1990

I certify that I have read this dissertation and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

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(Principal Adviser)

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Jussi Ketonen

I certify that I have read this dissertation and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

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Approved for the University Committee on Graduate Studies:

Dean of Graduate Studies

Abstract

Few systems for mechanical proof-checking have been used so far to *transform* formal proofs rather than to *formalize* informal arguments and to verify correctness. The *unwinding* of proofs, namely, the process of applying lemmata and extracting explicit values for the parameters within a proof, is an obvious candidate for mechanization. It corresponds to the procedures of Cut-elimination and functional interpretation in proof-theory and allows the extraction of the *constructive content* of a proof, sometimes yielding information useful in mathematics and in computing.

Resource-aware logics restrict the number of times an assumption may be used in a proof and are of interest for proof-checking not only in relation to their decidability or computational complexity, but also because they efficiently solve the *practical problem* of representing the *structure of relevance* in a derivation. In particular, in *Direct Logic* only one subformula-occurrence of the input is allowed, and the connections established during a successful proof-verification can be represented on the input without altering it. In addition, the values for the parameters obtained from unwinding are read off directly.

In *Linear Logic*, where classical logic is regarded as the limit of a resource-aware logic, long-standing issues in proof-theory have been successfully attacked. We are particularly interested in the system of *proof-nets* as a multiple-conclusion Natural Deduction system for Linear Logic.

In Part I of this thesis we present a new set of tools that provide a systematic and uniform approach to different resource-aware logics. In particular, we obtain uniqueness of the normal form for *Multiplicative and Additive Linear Logic* (sections 3 and 4) and an extension of Direct Logic of interest for nonmonotonic reasoning (section 8). In Part II we study Herbrand's Theorem in Linear Logic and the No Counterexample Interpretation in a fragment of Peano Arithmetic (section 10). As an application to Ramsey Theory we give a parametric form of the Ramsey Theorem, that generalizes the Infinite, the Finite and the Ramsey-Paris-Harrington Theorems for a fixed exponent (sections 10-13).

to Giulia and Guglielmo

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I would like to thank S. Feferman who has encouraged and supervised my work through all its different stages and rather unexpected changes of direction, and C. Talcott, who also has supported and contributed to this research in all stages. This thesis is a part of the project EKL, initiated and developed by J. Ketonen; I would particularly like to thank him for these years of stimulating and often exciting cooperation. The EKL project has been supervised by J. McCarthy, to whom also I want to express my gratitude.

It seems inappropriate for me to say how much my ideas have been shaped by those who are in my committee. In contrast, I can say that the approach and the teaching of D. Prawitz are still fresh in my mind, even after many years. It should be clear that the published works by J-Y.Girard (as well as his comments in conversation) have provided many of the topics and directions of the thesis. It should also be clear that the general direction of research has been inspired by G. Kreisel's philosophy, through his teaching and his challenging and stimulating correspondence. The results presented in part II are to be regarded as a first excursion into the territories he favors, an excursion I hope he does not entirely disapprove of.

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When I came to this country first, I was asked (in the same question) by the Department of Immigration whether I was infected with syphilis or carried illegal drugs or had been a member of 'certain organizations' — an expression which refers to the Italian parliamentary opposition, which at the time received about 30% of the votes. I confessed my flaw: in my youth I did not like the Party ruling on that

part of the world. Despite this, I was offered exciting opportunities of learning in this country and I want to thank the American taxpayers for having supported my intellectual curiosities for many years. This research has been supported by the NSF grant CCR-8718605 and Darpa contract N00039-84-C-0211.

Table of Contents

INTRODUCTION

1. Introduction.

It has often been remarked that a proof, in mathematics as well as in common sense reasoning, contains more information than the simple fact that its conclusion is true and serves purposes other than simply to convince us of its truth. In particular, a proof that some object exists which satisfies certain conditions often provides a computational method to construct such an object, or at least some indication of how it can be obtained when additional information is provided. Thus mathematical proofs are interesting for computation and the techniques of analysis of mathematical proofs, when applied to common sense reasoning, may lead to interesting applications in engineering.

In some sense proof theory — the study of logical inference — is the oldest branch of logic. The traditional questions '*What is a proof?*' and '*What kind of knowledge do we obtain through deductive reasoning?*' have been asked since the beginning of scientific thought. Since Euclid, the axiomatic method in mathematics has been the main paradigm of correct and successful deductive inference. Like the other branches of logic, proof theory owes its present form to the mathematical development of logical symbolism. Sharper questions are asked today; namely, '*When do two different formal derivations represent the same informal proof?*' and '*What information does a proof indirectly contain and how can it be made explicit?*' It would therefore be reasonable to expect that progress in proof theory would contribute to our understanding of mathematical reasoning, and that this in turn would lead to a genuine contribution by proof theory to the solution of mathematical problems.

Different strategies have been followed in the development of logic and of proof theory. Such differences have sometimes been regarded as arising from conflicting philosophical approaches and not simply from different styles of reasoning or mathematical tastes. Rather than discuss these philosophical issues directly here, we will provide a framework within which our technical results are to be understood and evaluated. Our strategy is to circumvent traditional controversies and to provide some results that are directly relevant to the theoretical issue of identity of proofs, and other results which contribute to the practical task of extracting implicit information from given mathematical proofs.

In this introduction, the problem of identity of proofs is considered in the light of Gentzen and Prawitz's fundamental contributions and of the more recent developments by Ketonen (Direct Logic) and Girard (Linear Logic). Our main technical results in this area will be presented and their significance indicated. Next, the problem of extracting implicit information from mathematical proofs is briefly discussed, and an application to Ramsey Theory is presented.

1.1. Features and Problems in Gentzen's Calculi.

Natural Deduction and Sequent Calculus. The lasting appeal of the contributions of G. Gentzen (Szabo [1969]) to proof theory seems to lie more in their phenomenological nature than in their foundational concerns. His system of Natural Deduction allows the introduction of assumptions and their removal (we speak of *open* and *closed* assumptions) and provides an *introduction rule* and an *elimination rule* for each logical symbol '$\circ$'. An introduction rule allows the formal inference of an expression with $\circ$ as the principal symbol ($\circ$-*expression*), and an elimination rule specifies how an $\circ$-expression can be used to infer other expressions. For instance, the introduction and elimination rules for $\wedge$ have the forms

$\wedge$ – **Introduction**	$\wedge$-**Eliminations**	
$\dfrac{A \quad B}{A \wedge B}$	$\dfrac{A \wedge B}{A}$	$\dfrac{A \wedge B}{B}$

The *Inversion Principle* is the requirement that for each $\circ$, the introduction and elimination rules for $\circ$ must be dual, in the sense that an introduction rule followed by an elimination rule yields one of the original premises. From a phenomenological point of view the analysis of informal reasoning through Natural Deduction is generally regarded as adequate. Prawitz [1965] made Natural Deduction a viable mathematical tool with his proof of the Normalization Theorem.

Gentzen [1935] had proved an analogous result (the *Hauptsatz* or the *Cut Elimination Theorem*) for his other creation, the system of Sequent Calculus. The calculus is based on sequences of expressions of the form $A_1, \ldots, A_m \vdash B_1, \ldots B_n$ (*sequents*), to be interpreted as formulas $\bigwedge_i A_i \supset \bigvee_j B_j$. Well-formed trees of sequents (*sequent derivations*) represent informal arguments. The expressions in the left part of a sequent (the *antecedent*) may be regarded as assumptions and those in the right part (the *succedent*) may be regarded as conclusions of the informal argument that is represented by the derivation ending with the sequent in question. In a sequent derivation some sequents (*axioms*) represent unanalyzed inferences. If S is not an axiom sequent, then S (*lower sequent*) is inferred from other sequents S_i (*upper sequents*), and in this case an inferential link may be established between a new occurrence of a formula (*principal formula*) in the antecedent or in the consequent of S and some occurrences of formulas in the sequents S_i (*active formulas*).

For each logical symbol $\circ$ there are $\circ$-*left* and $\circ$-*right* rules. A rule for $\circ$ allows the inference of a sequent whose principal formula is a $\circ$-expression and occurs in the antecedent (a left rule) or in the succedent (a right rule). For instance, the left and right rules for $\wedge$ have the forms

$$\begin{array}{ccc} \wedge - \textbf{Right} & & \wedge\textbf{-Left} \\ \dfrac{\Gamma \vdash \Delta, A \qquad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B} & \qquad & \dfrac{A, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta} \qquad \dfrac{B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta} \end{array}$$

(Here Γ and Δ stand for sequences of formulas.) A left rule corresponds to a step of an informal argument whereby given assumptions are inferred from a new assumption and a right rule corresponds to a step from given conclusions to a new conclusion. In addition, the lower sequent S contains all formulas of the upper sequents S_i different from the active formulas. Thus an inference in Sequent Calculus always updates the context of the assumptions and conclusions of the informal argument. Therefore, the right and the left $\circ$-rule of the Sequent Calculus correspond in Natural Deduction to the $\circ$-introduction and (the inversion of) the $\circ$-elimination rules, respectively, and in the Sequent Calculus the Inversion Principle is represented by the duality of the left and right rules for $\circ$.

In Natural Deduction the following deductive procedures can be represented: replacing an open assumption A of a deduction $\mathcal{D}_2$ with a deduction $\mathcal{D}_1$ of A (*substitution*); proving a more general lemma (*maximal formula*) and then applying it. The two procedures are different; only some substitutions have the effect of generating a new maximal formula in the resulting derivation. For instance, if

$$\begin{array}{c} \mathcal{D}_1 \\ A \end{array} \qquad\qquad \begin{array}{c} A \\ \mathcal{D}_2 \\ B \end{array}$$

are Natural Deduction derivations and the assumption A is open in $\mathcal{D}_2$ then substitution yields a deduction $\mathcal{D}$ of the form

$$\begin{array}{c} \mathcal{D}_1 \\ A \\ \mathcal{D}_2 \\ B \end{array}$$

However, A is a maximal formula in $\mathcal{D}$ only if it is the consequence of an introduction rule in $\mathcal{D}_1$ and the premise of an elimination rule in $\mathcal{D}_2$.

In Sequent Calculus, the operation of substitution is represented by the formal rule of Cut.

Cut:

$$\frac{\Gamma \vdash \Delta, A \qquad\qquad A, \Pi \vdash \Lambda}{\Gamma, \Pi \vdash \Delta, \Lambda}\ .$$

The active formulas in an application of this rule (*cut formulas*) play the same role as a maximal formula in a Natural Deduction derivation, except when one of the active formulas already occurs in a sequent axiom. Thus a procedure to eliminate the rule Cut from a system of Sequent Calculus corresponds to a procedure to eliminate maximal formulas in a corresponding system of Natural Deduction. A Natural Deduction derivation is *normal* if it contains no maximal formulas ("*All lemmata have been applied*"). A *Normal form theorem* for a Natural Deduction system for a certain logic asserts that every theorem of that logic can be derived with a normal proof in the system. A *Cut-Elimination Theorem* asserts that every theorem of a certain logic is provable in a Sequant Calculus for the logic in question, without the Cut rule. The existence of such a theorem for a certain logic depends of course on the *non-logical* axioms and rules of inference of the system. A Normalization (or Cut-Elimination) Theorem may sometimes be obtained by model-theoretic techniques. A proof-theoretic method usually provides (1) a set of *reduction steps*, one for each pair of introduction and elimination rules, and (2) a reduction procedure that always produces a suitable sequence of reduction steps yielding a normal proof (a *normalization theorem*) or that makes every sequence of reduction steps terminate (a *strong normalization theorem*).

Basic Results for the Gentzen-Prawitz Calculi. Natural Deduction for intuitionistic logic and Sequent Calculus for classical logic still remain the main success stories of proof theory. The use of proof-theoretic methods seems essential in these contexts. So far, model-theoretic methods have only provided weaker and less informative versions of the Hauptsatz.

From the preceding summary it may seem that a Natural Deduction and a Sequent Calculus for some logic are only minor variants of the same formal analysis

of intuitive reasoning. This view is too simplistic, as is clear from the facts indicated below. (*A detailed analysis of the relationships between the two* formalisms and in particular of the Cut-Elimination and Normalization Theorems is in Zucker [1974] and Ungar [1986].)

(1) In Natural Deduction the verification of the correctness of an inferential step may require a global examination of the deduction. In Sequent Calculus the verification of the correctness of an inferential step requires only local consideration of the sequents involved.

(2) In a system of Natural Deduction with rules for $\wedge$, $\vee$, $\supset$, $\forall$, $\exists$ and $\perp$ (*falsity*) formalizing intuitionistic first order logic, the Strong Normalization Theorem holds and the normal form is unique *modulo* permutations of the $\vee$ elimination and $\exists$ elimination rules. The Normalization Theorem also holds for higher order logic.

(3) In a Sequent Calculus with rules for $\neg$, $\wedge$, $\vee$, $\supset$, $\forall$ and $\exists$ formalizing classical higher order logic, the Cut Elimination theorem holds.

(4) A sequent calculus with rules for $\neg$, $\wedge$, $\vee$, $\supset$, $\forall$ and $\exists$ formalizing intuitionistic logic can be obtained by restricting the classical Sequent Calculus to sequents with at most one formula in the succedent.

In Sequent Calculus uniqueness of cut-free derivations is not an issue; in fact, a well-known result by Kleene [1952] gives rather general conditions that allow permutation of inferences in classical and intuitionistic Sequent Calculi. Using (2) and (4) for the fragment of intuitionistic logic without $\vee$ and $\exists$, we can define a many-to-one mapping π such that if $\mathcal{D}$ is a Sequent Calculus derivation of $B_1, \ldots, B_n \vdash A$, then $\pi(\mathcal{D})$ is a Natural Deduction derivation of A from $B_1, \ldots, B_n$.

Prawitz obtained a Natural Deduction representation of classical logic that essentially uses Gödel's double negation interpretation. He considers the fragment with rules for $\wedge$, $\supset$, $\forall$ and (a rule equivalent to) double negation elimination restricted to $\neg\neg A$ with A atomic. This representation is *not natural*. A classical derivation $\mathcal{D}$ of $B_1, \ldots, B_n \vdash A_1, \ldots, A_m$ can be represented by Natural Deduction derivations $\mathcal{D}_j$ of A_j from $B_1, \ldots, B_n$, $\neg A_1, \ldots, \neg A_{j-1}$, $\neg A_{j+1}, \ldots, \neg A_m$, for $j \leq m$. Thus the result of the previous paragraph is out of the question. More important

is the fact that the restriction on the double negation rule returns as a problem as soon we work with extensions of (this representation of) classical logic.[1]

It might seem that the distinctive properties of the systems of Natural Deduction and Sequent Calculus could capture the important structural distinctions between classical and intuitionistic logic. In particular, Prawitz and Dummett (see Dummett [1977]) noted that the introduction rules of Natural Deduction suggest an *operational* interpretation of the logical connectives very close to their *intuitionistic* meaning, and that such an interpretation seems impossible for the law of the excluded middle. They used this fact to argue in favor of Intuitionism versus Classical Logic. This foundational twist was already present in several passages by Gentzen himself.

However, the formal features of Gentzen-Prawitz's calculi might also suggest a different argument; namely, that in Natural Deduction as well as in Intuitionistic Logic we see a basic asymmetry between assumptions and conclusions, which is eliminated in Sequent Calculus and in Classical Logic. Such a line of thought was pursued by Girard (see Girard [1987b], [1988]). We believe that Girard's analysis significantly advances the program initiated by Gentzen and Prawitz. We will defend this claim in the first part of this dissertation.

1.1.1. An Overview of Direct and Linear Logic.

Multiple Conclusion Natural Deduction. A natural way to extend the mapping π from intuitionistic to classical logic would be to provide a system of Natural Deduction in which the rules are allowed to have multiple conclusions. Extensive efforts in this direction (see Ungar [1986]) have not yet brought conclusive results. To bring this problem closer to a solution, fragments of classical propositional logic have to be considered first.

[1] As an example, consider the Normalization Theorem for a Natural Deduction system for GL, the modal system for the logic of provability (see Bellin [1985]). Here the Natural Deduction framework permits an elegant representation of the main diagonalization property of this logic — namely, if we have a derivation of A from a class of assumptions of the form $\Box A$, then we can conclude $\Box A$ and discharge the class of assumptions in question. However, the double negation elimination, applied to $\neg\neg\Box A$, requires massive transformations in the proof of normalization.

Relevance Logics. Rather puzzling results came from Relevance Logic, a traditional family of subsystems of classical logic sharing the property that their Sequent Calculus does not admit the Weakening rule.

Weakening:

$$\frac{\vdash \Gamma}{\vdash \Gamma, A}$$

In 1984 Urquhart proved that propositional relevance logic is undecidable. However, the system of Sequent Calculus for propositional relevance logic with the Mingle rule

Mingle:

$$\frac{\vdash \Gamma \qquad \vdash \Delta}{\vdash \Gamma, \Delta}$$

is decidable.

Direct Logic. From our point of view the most important contribution arrived from another direction, namely from logics that are *resource aware* — shall we speak of *ecological* logics? — i.e., logics whose systems of Sequent Calculi do not admit the Contraction rule.

Contraction:

$$\frac{\vdash \Gamma, A, A}{\vdash \Gamma, A}$$

It is well known that a Sequent Calculus for first order logic without the Contraction rule formalizes a *decidable* fragment, called *Direct Logic* in Ketonen and Weyhrauch [1984]. A decision procedure is simply the application of the rules of Sequent Calculus in inverse order (*Wang algorithm*, see Wang [1960]). Such a method was presented first by Oiva Ketonen in his thesis [1944]). In the absence of the Contraction rule, for each essentially existential quantifier[2] we are allowed to try only one substitution, and the search procedure soon reduces to the propositional case.

[2] A quantifier and its bound variable are *essentially universal* if they are universal in a positive context or existential in a negative context; otherwise, they are *essentially existential*.

The original motivation for Ketonen and Weyhrauch [1984] was to provide a less bureaucratic, more natural procedure than the Wang algorithm and to implement it in the LISP-based Proof Checker EKL.

> "The very locality of the rules of sequent calculus forces us to neglect the global 'leitmotifs' of proofs. For example, the mechanical application of inverted two-premise rules not only duplicates the work to be done in every succeeding step but may also be unnecessary. The 'global picture', i.e., the natural order of application of inference rules, cannot be deciphered through the application of this kind of formalism. The problems presented by the classical resolution approach are similar: again we are faced with the necessity of a global, conjunctive normal form, a transformation that erases the local connections used in natural proof generation." (J.Ketonen, manuscript)

By allowing global consideration of the data, a method is designed that splits the current data into two independent parts at each application of an inverted two-premise rule. For instance, in the Wang algorithm we invert a rule for conjunction of the form

Additive Conjunction:

$$\frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B}$$

while the method of Ketonen and Weyhrauch allows us to invert a rule of the form

Multiplicative Conjunction:

$$\frac{\vdash \Gamma, A \qquad \vdash B, \Delta}{\vdash \Gamma, A \wedge B, \Delta} \quad .$$

The method is essentially a search for *adequate* pairings of negative and positive occurrences of atomic formulas, corresponding to the axioms of a derivation. The condition for adequacy must consider the global connections determined by the atomic pairings (*atom links*), and by the formulas corresponding to conclusions of two-premise rules (*two-premise links*). The key notion here is that of a *chain*, a data-structure determined by the alternation of these two kinds of connections. The main requirement is that no such chain may be *cyclic* (see section 5.1). Using the

Cut Elimination theorem for Sequent Calculus and the absence of Contraction, we may work with the subformulas of the given set of formulas, each subformula being considered just once. These data have a natural representation as LISP structures. Ideas of Andrews [1981] and of others are used or independently rediscovered here. (Another analysis of Contraction-free logics can be found, e.g., in H. Ono and Y. Komori [1985].) The main theorem claims that success in the search is equivalent to provability in the Sequent Calculus for first order Direct logic. A proof for the propositional case can be found in sections (8.3)-(8.8) below.

It is significant that irrelevance in this context represents a practical problem — how to detect it and how to reduce the scope of the search efficiently — rather than a theoretical evil that knocks the search off as soon as its uneliminable presence is detected. However, in first order logic the claim holds for prenex formulas only. In fact, Herbrand's Theorem is used in Ketonen and Weyhrauch [1984] to reduce the first order case to the propositional case. But Herbrand's Theorem fails for non-prenex formulas, as shown by the following counterexample. If B does not contain x free, then $\big(\forall x.(A(x) \land B)\big) \supset (\forall y.A(y)) \land B$ does not hold in Direct Logic but the Herbrand expansion $A(\mathbf{y}) \land B \supset A(\mathbf{y}) \land B$ holds in Direct Logic. A proof procedure may use classical logic to transform formulas into prenex normal form in classical logic and then apply the technique of Ketonen and Weyhrauch [1984]. Such a procedure violates the main motivations for Direct Logic presented above; in fact the transformation of formulas into prenex form alters the correspondence between the formal syntax and the intuitive meaning and therefore destroys the 'natural order of application of the inference rules' implicit in the original form of the formulas. It is therefore fair to say that Direct Logic remains an unfinished project. The full first order case can now be obtained using ideas from Linear Logic, as discussed below.

Linear Logic. Girard [1987b] suggests that a major motivation for Linear Logic comes from the theory of models for Lambda Calculus and presents the evolution from *Scott's domains* to *qualitative domains* and to *coherent spaces* as its appropriate genealogy. It is interesting to note that although coherent spaces provide a Heyting-style interpretation for Linear Logic, the interpretation is not faithful. In Linear Logic there are constants $\mathbf{1}$, $\perp$, $\top$ and $\mathbf{0}$; regarded as propositions, $\perp \neq \mathbf{1}$ and

$\mathbf{0} \neq \top$, while regarded as coherent spaces, $\bot = \mathbf{1}$ and $\mathbf{0} = \top$. In fact all other techniques presented in the paper are readable in an interesting way from the point of view of proof theory alone. Here we will try to motivate the laws of Linear Logic starting from the Sequent Calculus presented in Girard [1987b]. The rules of the calculus are listed in section (2.1).

The main idea is the introduction of an operator '!' (with dual '?'). The fragment without these operators is free from irrelevance and accountable with respect to resources, while in the full system this kind of information may be lost.

The operator ! has the same rules as the □ operator in the modal system S4, namely

! Rule: **Dereliction:**

$$\frac{\vdash ?C_1, \ldots, ?C_m, A}{\vdash ?C_1, \ldots, ?C_m, !A} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A} \quad .$$

In addition, Weakening and Contraction are allowed for formulas of the form $?B$.

Stronger operations, the *linear implication* $\multimap$ and the involutory *linear negation* $(.)^\perp$, are introduced. Classical or intuitionistic implication $A \supset B$ is represented as $!A \multimap B$ and classical negation $\neg A$ is represented as $(!A)^\perp$. In the tradition of Heyting's semantics, the constructive meaning of $A \supset B$ is a method that transforms a proof of A into a proof of B. The constructive meaning of $A \multimap B$ is a method that uses a proof of A exactly once in producing a proof of B.

The distinction is made between *additive* and *multiplicative* conjunctions and disjunctions, a restatement of the distinction between *extensional* and *intensional* connectives in Relevance Logic. To guarantee duality between conjunction and disjunction we must have

Additive Disjunction

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, A \vee B} \qquad \frac{\vdash \Gamma, B}{\vdash \Gamma, A \vee B}$$

and

Multiplicative Disjunction

$$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B} \quad .$$

In conclusion, the language of Linear Logic contains the logical symbols in the following table.

Linear Negation: $(.)^{\perp}$	
Multiplicatives:	
Conjunction: $\otimes$ (*times*)	*Disjunction*: $\sqcup$ (*par*)
with identity $\mathbf{1}$	with identity $\perp$
Additives:	
Conjunction: $\sqcap$ (*with*)	*Disjunction*: $\oplus$ (*plus*)
with identity $\top$	with identity $\mathbf{0}$
Quantifiers:	
$\bigwedge x$ (*for all* x)	$\bigvee x$ (*there exists* x)
Exponentials:	
! (*of course!*)	? (*why not?*)

Linear Logic can be *Commutative* or *Noncommutative*, depending on the validity or not of the axioms

$$A \otimes B = B \otimes A, \qquad A \sqcup B = B \sqcup A, \qquad B \circ\!\!- A = A -\!\!\circ B.$$

In Noncommutative Linear Logic formulas of the form $\perp$ and $?A$ commute with everything.

Translations into Linear Logic.

Intuitionistic and classical logic have standard translations $(.)^0$ and $(.)^+$ into Linear Logic (see Girard [1987b]), presented in the table below. The fundamental asymmetry between assumptions and conclusions in intuitionistic logic is represented by the asymmetry of the exponentials ! and ?.

A combinatorial motivation for the translation $(.)^+$ of Classical Logic can be suggested through the following application of the Cut rule in classical Sequent Calculus:

$$\begin{array}{lcc}
S_1: & \Gamma \vdash \Delta, A, A & S_3: \quad A, A, \Pi \vdash \Lambda \\
\cline{2-2} \cline{3-3}
S_2: & \Gamma \vdash \Delta, A & S_4: \quad A, \Pi \vdash \Lambda \\
\hline
S_5: & \multicolumn{2}{c}{\Gamma, \Pi \vdash \Delta, \Lambda}
\end{array}.$$

A standard reduction step in the Cut elimination procedure involves *cross-cutting*:

(1) consider a Cut with premises S_1 and S_4 and apply the cut elimination procedure to obtain S': $\Gamma, \Pi \vdash \Delta, \Lambda, A$;

(2) consider a Cut with premises S_2 and S_3 and apply the cut elimination procedure to S'': $A, \Gamma, \Pi \vdash \Delta, \Lambda$;

(3) apply Cut to S' and S'', yielding $\Gamma, \Gamma, \Pi, \Pi \vdash \Delta, \Delta, \Lambda, \Lambda$;

(4) apply Contraction several times to obtain S_5.

Clearly this procedure is not ecological and produces a combinatorial explosion of the data, since Contraction is repeatedly applied — in steps (1), (2) and (4). Consider the following classical derivation:

$$\dfrac{\dfrac{\dfrac{\vdash P, \neg P \qquad \dfrac{\vdash P, \neg P \qquad \vdash Q, \neg Q}{\vdash P, \neg P \wedge \neg Q, Q}}{\vdash \neg P \wedge \neg Q, \neg P \wedge Q, P, P}}{\vdash \neg P \wedge \neg Q, \neg P \wedge Q, P} \qquad \dfrac{\dfrac{\vdash \neg P, P \qquad \dfrac{\vdash \neg P, P \qquad \vdash \neg Q, Q}{\vdash \neg P, P \wedge \neg Q, Q}}{\vdash \neg P, \neg P, P \wedge \neg Q, P \wedge Q}}{\vdash \neg P, P \wedge \neg Q, P \wedge Q}}{\vdash \neg P \wedge Q, \neg P \wedge \neg Q, P \wedge \neg Q, P \wedge Q}.$$

This clearly fails in Linear Logic without ? and !. Since in the full language we have Contraction only for formulas of the form $?B$, the above proof will be representable in Linear Logic only if the cut formula is $?!P$. In this representation the formulas responsible for combinatorial explosions in the normalization procedure are readily identified.

Translation of *Intuitionistic Logic*:

$$(P)^0 = P, \text{ for } P \text{ atomic;}$$

$$(A \wedge B)^0 = A^0 \sqcap B^0, \qquad (A \vee B)^0 = !A^0 \oplus !B^0$$

$$(A \supset B)^0 = (!A^0) \multimap B^0, \qquad (\neg A)^0 = (!A^0) \multimap 0$$

$$(\forall x.A)^0 = \bigwedge x.A^0, \qquad (\exists x.A)^0 = \bigvee x.(!A^0).$$

Translation of *Classical Logic*:

$$(P)^+ = ?!P, \text{ for } P \text{ atomic;}$$

$$(A \vee B)^+ = A^+ \sqcup B^+, \qquad (\neg A)^+ = ?A^{+\perp}$$

$$(\forall x.A)^+ = ?!\bigwedge x.A^+.$$

1.1.2. Sequent Calculus and Natural Deduction for Linear Logic.

Multiple Conclusion Natural Deduction for Linear Logic. We reconsider the issue of Sequent Calculus and Natural Deduction in the context of Linear Logic. The question is: *Can a satisfactory form of Multiple Conclusion Natural Deduction be provided for Linear Logic?* We claim that the answer is provided by a reformulation of the system of ***proof nets*** introduced for the multiplicative fragment in Girard [1987b]. We simplify that system, extend it to the additive fragment and describe a possible extension to the whole system.

We consider first some differences between Sequent Calculus and Natural Deduction. In Natural Deduction

(1) *assumptions* can be made and sets of assumptions *discharged* or *closed*, and an inferential step consists of making an assertion (*conclusion*) on the basis of some assertions (*premises*) previously made;

(2) the determination of which assumptions are open or closed at each inferential step is in accordance with general *deduction rules* or depends on *ad hoc* specifications;

(3) some inferential steps introduce a statement which is more complex than the premises (*introduction rules*) and other steps use a complex statement to infer a simpler one (*elimination rules*).

A Sequent Calculus derivation represents an informal argument as follows:

(1) *axiom sequents* are taken as unanalyzed inferences and a sequent may be inferred from other sequents;

(2) special *structural rules* of the calculus are used for the administration of the expressions in the antecedent and in the consequent;

(3) if a sequent S is inferred from other sequents S_i, the expressions in S_i are either the same as or subexpressions of the expressions in S.

Right only, Introduction only Systems. The same principle of duality (Inversion Principle) that governs the relations between introduction and elimination rules in Natural Deduction and between Right and Left rules in Sequent Calculus holds between the Right rules of dual connectives or quantifiers in the Sequent Calculus for Linear Logic. Thus notational economy is achieved — at the expense of some intuitive discomfort — by working with sequents consisting only of the succedent part in a calculus containing only Right rules (see section 2.1). Similarly, we may have a Natural Deduction with only introduction rules, with *multiple conclusion axiom links* of the form

$$\overline{P}, \overline{P^{\perp}}.$$

In these formalisms, the open assumptions of an intuitive argument will therefore appear as negated conclusions. In conclusion, in a Multiple Conclusion Natural Deduction system the premises of each link are subformulas of the conclusion, and difference (3) between the calculi disappears.

Closure of open assumptions. The rules of Gentzen-Prawitz's Natural Deduction related to the closure of open assumptions are $\supset$-introduction, $\vee$-elimination, $\exists$-elimination, and the rule corresponding to double negation elimination. Now $\sqcup$-elimination is represented by a $\otimes$-*link*, $\oplus$-elimination by a $\sqcap$-*link*, and $\bigvee$-elimination by a $\bigwedge$-*link*. Since $(.)^{\perp}$ is involutory, we may let $A^{\perp\perp}$ be A by definition. It would be natural to regard the $\multimap$-*link* as a two-premise rule, the second premise being the unique assumption discharged at the inference. Further notational economy

is achieved by regarding $\multimap$ as defined, thus a $\multimap$-link with conclusion $A \multimap B$ is represented by the $\sqcup$-link:

$$\frac{A^{\perp} \quad B}{A^{\perp} \sqcup B}$$

In conclusion, the mechanism for discharging assumptions becomes unnecessary — or perhaps we should say it is disguised under the $\otimes$-, $\sqcap$- and $\sqcup$-links — and difference (1) between the formalisms is obliterated. However, we should not be surprised that the distinction between open and closed assumptions still plays a role in the Multiple Conclusion Natural Deduction System, as we will discover when considering the *doors* of an *empire* (see section 3.4).

Structural Rules for Natural Deduction? Finally, it would seem that structural rules should explicitly belong to a Natural Deduction formulation of Linear Logic. We will provide a formulation of *Proof-Structures for Linear Logic* which contains some explicit forms of structural rules (see section 3) and is therefore similar to Sequent Calculus with respect to (3). However we must be careful, since structural rules are a major obstacle to the uniqueness of the normal form.

We focus on the multiplicative and additive fragment **MALL**. In the Sequent Calculus for **MALL**, *Weakening* is admitted only to introduce the propositional constant $\perp$

$\perp$-Rule:

$$\frac{\vdash \Gamma}{\vdash \Gamma, \perp}$$

and in the axiom for $\top$

$$\vdash \top, \Gamma.$$

Contraction is implicit in the rule for $\sqcap$:

$\sqcap$-Rule:

$$\frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \sqcap B} \ .$$

A possible translation of this rule into Natural Deduction is the introduction of the inferential links $\frac{A \quad B}{A \sqcap B}$ and a sequence of Contraction links $\frac{C \quad C}{C}$ for each C occurring in Γ (*hidden contractions*). The use of Contraction links must be restricted,

and each proof-structure S is equipped with a *flagging function* φ that associates each Contraction link with a $\sqcap$ link.

We suspect that the only plausible solution to the problem of translating Weakening is the introduction of its principal formulas among the axioms. This is Girard's formulation in the case of $\top$. Nothing is lost if $\perp$ is introduced in any axiom of the derivation above $\vdash \Gamma$, rather than by the $\perp$ Rule, unless a special meaning is attributed to the fact that an axiom link $\overline{P, P^{\perp}}$ consists of *two* occurrences. In this case the Weakening rule in general becomes

$$\frac{\vdash \Gamma, \perp}{\vdash \Gamma, ?A.}$$

Finally, the rule of *Cut* is represented by a *cut* link. This is an instance of a $\otimes$-introduction with a conclusion of the form $A \otimes A^{\perp}$. However, such a $A \otimes A^{\perp}$ *cannot* be a premise of another link and is *not* regarded as a conclusion of the deduction. For this reason we underline $A \otimes A^{\perp}$, when it is regarded as conclusion of a cut. We want to emphasize the following point: with respect to the internal connections in a deduction, a Cut link is exactly the same as a $\otimes$-link. However, in the external consideration of a derivation as yielding certain conclusions, an underlined formula-occurrence must be regarded as a local and temporary contradiction, that does not belong to the conclusions of the derivation.

Cut:

$$\frac{A \qquad A^{\perp}}{\underline{A \otimes A^{\perp}}}$$

Families of Quasi-Structures. However, the hidden Contraction is damaging for the uniqueness of the normal form, since it allows the permutation of the order of inferences $\sqcap/\otimes$, as shown in

$$\frac{\vdash \Delta, D \qquad \dfrac{\vdash C, \Gamma, A \qquad \vdash C, \Gamma, B}{\vdash C, \Gamma, A \sqcap B}}{\vdash \Delta, D \otimes C, \Gamma, A \sqcap B}$$

and in

$$\frac{\dfrac{\vdash \Delta, D \qquad \vdash C, \Gamma, A}{\vdash \Delta, D \otimes C, \Gamma, A} \qquad \dfrac{\vdash \Delta, D \qquad \vdash C, \Gamma, B}{\vdash \Delta, D \otimes C, \Gamma, B}}{\vdash \Delta, D \otimes C, \Gamma, A \sqcap B} \, .$$

An idea for the elimination of the hidden contraction rule is the theory of '*slices*' suggested in chapter 6 of Girard [1987b], pp.93-97:

> "The slicing is the 'development' of a proof-net by 'distributivity'. Each slice is itself logically incorrect, but it is expected that the total family of slices has a logical meaning. However, there is no characterization of the set of slices which are the slices of a proof-net. Moreover, we do not even know whether or not identifying proof-nets with the same slices could lead to logical atrocities..." (Remark 6.3.(i), page 95)

In section (6.1) we develop this notion. We define *Families of Quasi-Structures for Linear Logic* and we show that no logical atrocity results if we consider a slicing as a set of *embeddings* of a family of quasi-structures into a given proof-structure. The families of quasi-structures representing valid proofs will be called *Proof Networks*. In section (6.3) we prove an Equivalence Theorem between Proof-Nets and Proof-Networks. As a Corollary, two derivations that differ only for a permutation of inferences $\sqcap/\otimes$ are projected into the same proof-network. This solves the problem of the uniqueness of the normal form, at least for this fragment of Linear Logic.

Global vs. Local verification; Natural Deduction in Sequent-like notation. From the analysis summarized above it appears that the main difference between the Multiple Conclusion Natural Deduction and the Sequent Calculus for Linear Logic is the *global* versus *local* character of the verification of correctness of each inferential step, as described in section (1.1), fact (1). This point deserves a more detailed discussion.

The verification of correctness in Gentzen-Prawitz's Natural Deduction may depend on global conditions; still, we can define a (rather bureaucratic) system of accounting of the open assumptions so that the necessary information is available at each step of the inductive construction a derivation. Such a system, the *sequent*

formulation of Natural Deduction, was introduced in Gentzen [1936]. We may write

Assumptions:

$$(A) \vdash A$$

$\supset$ –Introduction

$$\frac{(\Gamma, A) \vdash B}{(\Gamma) \vdash A \supset B}$$

$\supset$ –Elimination

$$\frac{(\Gamma) \vdash A \supset B \qquad (\Delta) \vdash A}{(\Gamma, \Delta) \vdash B}$$

$\forall$ – Introduction

$$\frac{(\Gamma) \vdash A(a)}{(\Gamma) \vdash \forall x.A(x)}$$

$\forall$ – Elimination

$$\frac{(\Gamma) \vdash \forall x.A}{(\Gamma) \vdash A(t)}$$

where a does not occur free in Γ,

Weakening

$$\frac{(\Gamma) \vdash B}{(\Gamma, A) \vdash B}$$

Contraction

$$\frac{(\Gamma, A, A) \vdash B}{(\Gamma, A) \vdash B} \ .$$

This is still a Natural Deduction system (for a fragment of intuitionistic logic), where the expressions in parentheses on the left of $\vdash$ are only *annotations* and the verification of correctness for the $\forall$-rule is still global. In particular, if

$$\begin{array}{c} \mathcal{D}_1 \\ (\Delta) \vdash A \end{array} \qquad \begin{array}{c} (A) \vdash A \\ \mathcal{D}_2 \\ (\Gamma, A) \vdash B \end{array}$$

are derivations and

$$\begin{array}{c} \mathcal{D}_1 \\ (\Delta) \vdash A \\ \mathcal{D}_2 \\ (\Gamma, \Delta) \vdash B \end{array}$$

results from $\mathcal{D}_1$ and $\mathcal{D}_2$ by substitution, then the annotations have to be changed and A is replaced by Δ throughout $\mathcal{D}_2$. In other words, the administration of the general context cannot be dealt with locally.

Girard's System of Proof-Nets. Girard gives a condition to distinguish the proof structures in the multiplicative fragment without constants that represent correct

derivations (*Proof-Nets*) from those which do not. This condition results from certain searches through a proof structure and requires that all possible searches should cross all formulas in the structure in two different directions before returning to the starting point in the same direction (*no-short-trip condition*, see Girard [1987b], 2.4., page 33). However, in the proof of the *Sequentialization Theorem*, which allows recovery of a sequent derivation corresponding to a given proof-net, the main notion is that of *empire*, which is defined there in terms of the search procedure (see Girard [1987b], 2.9.3., p. 37; see also section (3.4) below. The role of the notion of empire is even more obvious in the alternative proof sketched in Girard [1987c], II.1., remark 2). Therefore it seems natural that Girard's result can be simplified if empires are defined directly rather than in terms of all possible searches across a structure.

1.1.3. A Review of Results.

The main subsystems of Linear Logic under consideration are **MLL** (*Multiplicative Linear Logic*) and **MALL** (*Multiplicative and Additive Linear Logic*), in their propositional and first order versions. **NCMLL** and **NCMALL** are the respective Non Commutative versions.

Proof-Structures. *Proof-Structure* are defined in section (3) as sets of *formula-occurrences* satisfying certain relations, the *links*. In sections (3.1) and (3.2) we define a subclass of such graphs, the *inductively defined* proof-structures, together with a list of *conclusions*, the lowermost formula-occurrences in the graph. The inductive definition starts with a proof-structure consisting of an *axiom link* and then adds new links between the conclusions of already defined proof-structures. In the inductive definition of a proof-structure for propositional **MALL** the following restrictions are introduced:

(i) a $\otimes$-link $\frac{A \quad B}{A \otimes B}$ can be introduced only if A and B are conclusions of different proof structures $\mathcal{S}_1$ and $\mathcal{S}_2$, where $\mathcal{S}_1$ and $\mathcal{S}_2$ share no formula-occurrence.

(ii) a $\sqcup$-link $\frac{A \quad B}{A \sqcup B}$ can be introduced only if A and B belong to the same proof-structure.

(iii) a $\sqcap$-link $\frac{A \quad B}{A \sqcap B}$ can be introduced only if A and B belong to different proof structures $\mathcal{S}_1$ with conclusion $C_1, \ldots, C_n, A$ and $\mathcal{S}_2$ with conclusion $C_1, \ldots, C_n, B$

such that $\mathcal{S}_1$ and $\mathcal{S}_2$ share no formula-occurrence and for all $i \leq n$ a contraction link $\frac{C_i \quad C_i}{C_i}$ is also introduced.

We will see below (sections 3.6, 3.7) that *all inductively defined proof-structures are proof-nets* and naturally correspond to sequent derivations. This is hardly a surprise, since the inductive definition of the proof-structure already guarantees the conditions on the context that in Sequent Calculus are checked through the passive formulas of the sequent.

Thus the nontrivial task is to provide (algebraic or geometric) conditions that allow us to recognize whether a given proof-structure can be generated inductively. Moreover, such conditions should be relatively simple when compared to the formalism of sequent calculus and its allegedly excessive 'bureaucracy'. We believe that a concise system of annotations for proof-nets can be given, and that the restrictions introduced by such annotations can be intuitively justified — see sections (1.1.5) and (4.7).

Empires and Relevance. The crucial notion of the *empire* $e(A)$ of a formula-occurrence A in a proof-structure $\mathcal{S}$ is inductively defined in section (3.4). Membership in $e(A)$ propagates to the subformulas and through axiom links. We know from Girard [1987b], (see facts 2.9.4. p.37) that if A is a formula occurrence different from B and C, we have two cases:

(1) *if* $B \in e(A)$ *then* $B \otimes X$ *or* $X \otimes B \in e(A)$, *for all* X;

(2) *if* $B \in e(A)$ *and* $C \in e(A)$ *then* $B \sqcup C \in e(A)$.

To extend this approach to the additives we must take

(1) *if* $B \in e(A)$ *then* $B \oplus X$ *or* $X \oplus B \in e(A)$, *for all* X;

(2) *if* $B \in e(A)$ *and* $C \in e(A)$ *then* $B \sqcap C \in e(A)$.

The empire of A in $\mathcal{S}$ can be described as the substructure of $\mathcal{S}$ which is *deductively relevant* to A.

The lowermost formula-occurrences in $e(A)$ are called the *doors* of $e(A)$. These are either A itself (*main door*) or conclusions of $\mathcal{S}$ (*fake doors*) or premises D of a link $\frac{C \quad D}{C \sqcup D}$, $\frac{C \quad D}{C \sqcap D}$ or of a contraction-link $\frac{D, D'}{D''}$ such that C and D' do not belong to $e(A)$ (*side doors*). It would be perhaps more instructive to speak of *open* and *closed* doors rather than *fake* and *side* doors. Finally, a conclusion of a Cut link may

be called a *ghost* door, one that can never be used as a door and may be removed altogether.

Proof-Nets. The conditions characterizing propositional proof-nets given in section (3.5) are:

(i) for all conclusions $A \otimes B$ of a $\otimes$-link in $\mathcal{S}_\varphi$, $B \notin e(A)$ and $A \notin e(B)$ (*absence of vicious circles*);

(ii) for all conclusions $A \sqcup B$ of a $\sqcup$-link in $\mathcal{S}_\varphi$ and for all pairs A, B of conclusions of $\mathcal{S}$, $A \in e(B)$ and $B \in e(A)$ (*connectedness*);

(iii) for all conclusions $A \sqcap B$ of a $\sqcap$-link in $\mathcal{S}_\varphi$, there is no open door of $e(A)$ nor of $e(B)$ and the closed doors of $e(A)$ and of $e(B)$ are all premises of contraction links associated with the $\sqcap$-link in question by the function φ. (*enclosure in a 'box'*).

By the Box condition, if a Contraction link

$$\frac{D' \quad D''}{D},$$

is associated with

$$\frac{A \quad B}{A \sqcap B},$$

then D' is a door of $e(A)$ and D'' is a door of $e(B)$, and all the side doors of $e(A)$ and $e(B)$ are of this form.

For first-order proof-nets we need an additional condition, adapted from Girard [1987c]. Given a link $\frac{A(a)}{\bigwedge x.A(x)}$, the variable a is called an *eigenvariable* or a *parameter*. We assume that every eigenvariable is associated with a unique $\bigwedge$-link. We define a relation $a <^0 b$ between eigenvariables as follows. If a is associated with $\frac{A(a)}{\bigwedge x.A(x)}$, then we let $a <^0 b$ if b occurs inside and outside $e(A(a))$. Let $<_t$ be the transitive closure of $<^0$. The additional condition for first order proof-nets is:

(iv) the relation $<_t$ is a strict partial ordering (*parameters condition*).

Structure Theorem and Sequentialization Theorem. It is easy to see that all inductively defined proof structures satisfy these conditions (section 3.6). It is easy to define a mapping π from Sequent Calculus derivations to proof-structures and it is also easy to show that the image of any π is inductively defined (*Structure*

Theorem). This proves that π is a (many-to-one) mapping into the set of proof-nets (section 4).

To show that π is onto (*Sequentialization Theorem*), we need some additional facts about empires. Searches through empires are formalized as trees (section 4.1), a simple instance of a general notion of inductive search extensively studied by Feferman [1982]. The main result (*Tiling Lemma*, section 4.3) is that for all A, B in $\mathcal{S}$ the only cases besides $e(A) \cap e(B) = \emptyset$ and $e(A) = e(B)$ are:

(i) $A \in e(B)$ and $B \notin e(A)$ (notation: $A \sqsubset B$), which implies $e(A) \subsetneq e(B)$;

(ii) $A \notin e(B)$ and $B \in e(A)$ (notation: $B \sqsubset A$), which implies $e(B) \subsetneq e(A)$;

(iii) $A \in e(B)$ and $B \in e(A)$ (notation: $A \Diamond B$).

Another significant use of the notion of empire is the *Substructure Theorem*: for all A, $e(A)$ is a substructure of $\mathcal{S}$ which is also a proof-net.

With this preparation, the proof of the Sequentialization Theorem for **MLL** follows the 'alternative proof' in Girard [1987c] II.1., remark 2 (section 4.4.1). The difficult case in the argument is that of a proof-net $\mathcal{S}$ in which all conclusions have the form $A_i \otimes B_i$, for $i \leq n$. For such an $\mathcal{S}$ we need to find a $j \leq n$ such that $e(A_j) \cup e(B_j) \cup \{A_j \otimes B_j\}$ is $\mathcal{S}$. We argue that if $A_1 \otimes B_1$ does not have this property, then there must be a side door D of $e(A_1)$ and a conclusion $A_i \otimes B_i$ such that D is a subformula of $A_i \otimes B_i$, say of A_i. Since the empire of a formula X contains the subformulas of X, we have $e(A_1) \cap e(A_i) \neq \emptyset$ and $A_i \notin e(A_1)$, thus $e(A_1) \subsetneq e(A_i)$. Also we can show that $e(B_1) \subsetneq e(A_i)$. We proceed in this way until we find an $A_j \otimes B_j$ where $e(A_j)$ or $e(B_j)$ is maximal with respect to $\subsetneq$.

In the case of **MALL** we define a relation '$\triangleleft$' between formula-occurrences of the form $A \sqcap B$ by setting $A' \sqcap B' \triangleleft A \sqcap B$ if $A' \sqcap B'$ occurs in $e(A)$ or in $e(B)$. We show that $\triangleleft$ is a strict partial ordering (section 4.5).

For the first order case Girard [1987c] presents a condition on all possible searches through a proof-structure and derives the parameters condition from it by a difficult argument. In our exposition we achieve a drastic simplification; in order to prove the Sequentialization Theorem we need only the following easy fact. Let a, b be eigenvariables, where a is associated with $\forall x.A(x)$. If $b = a$ or $b <_t a$ in a proof-net, then b occurs only inside $e(A(a))$ (see the proposition in section 4.6).

Chains and Cycles. A direct analysis of empires does not seem to lead to an elegant proof of the Cut-elimination Theorem. It is convenient to use in this context the notions of a *path*, a *chain* and a *cycle* introduced by Ketonen and Weyhrauch [1984] (see section 5.1). Given a set of formula-occurrences Γ, a proof-structure is completely determined by a set of axiom links between their (atomic) formulas is established — the proof-structure containing all the subformulas of Γ with the specified axiom links. A *path* for Γ is simply a set of axiom links satisfying certain conditions and a *chain* is a global connection, determined by the axiom-links and the subformulas of the form $A \otimes B$. *If we assume the connectedness condition*, then the absence of *cycles* in the sense of Ketonen and Weyhrauch [1984], namely, of cyclic chains, is equivalent to the absence of vicious circles. By this method the Strong Normalization Theorem for **MLL** is easily proved.

Families of Quasi-Structures and Uniqueness of Normal Form. However, we aim at a Strong Normalization Theorem for **MALL** together with the uniqueness of the normal form. Our discussion in section (1.1.2) shows that uniqueness of the normal form is impossible unless we neutralize the hidden contractions of the $\sqcap$-rule. As pointed out above, this can be achieved through the notion of a family of *quasi-structure* together with their embeddings into the given proof-structure. In a quasi-structure, $\sqcap$ links are unary and there are no Contraction links. An embedding of quasi-structures into a proof-structure must preserve the links (*modulo* the iterations of formula-occurrences in the Contraction links of a proof-structure). A *family of quasi-structures Fam* S_φ associated with a given proof-structure S_φ is the set of its maximal subsets (in a suitable sense) that are quasi-structures. (see section 6). We may regard structures as a category **Struct**, containing both proof-structures and quasi-structures as objects, with embeddings as morphisms. Then the operation "*connect two disjoint proof-structures with a $\otimes$-link*" is a functor (*Times*) $\otimes$: $\mathbf{Struct}^* \times \mathbf{Struct}^* \to \mathbf{Struct}^*$, where $\mathbf{Struct}^*$ is the category of Structures with a selected conclusion.

An interesting result is the generalization of the notion of empire (section 6.2). For each A in a given proof structure, the set of embeddings in question induces an equivalence class $[A]$ of formula-occurrences in the corresponding family of quasi-structures. We say that the equivalence class $[B]$ belongs to the empire of an

equivalence class $[A]$ (notation: $[B] \in e[A]$), if for all Q such that a representative B_Q of $[B]$ occurs in Q, there is a representative A_Q of $[A]$ in Q and $B_Q \in e(A_Q)$.

The propositional conditions characterizing a proof-network $\mathcal{F} = Fam\ \mathcal{S}_\varphi$ are:

(i) the absence of vicious circles *in every quasi-structure* of $\mathcal{F}$;

(ii) the connectedness *in every quasi-structure* of $\mathcal{F}$;

(iii) the axioms of each quasi-structure in $\mathcal{F}$ are already axioms of $\mathcal{S}_\varphi$, and if D and D' are associated with the same premise of a $\sqcap$-link, then they are always connected in $\mathcal{F}$ — more precisely, if $\varphi(D) = \varphi(D')$ in $\mathcal{S}_\varphi$, then $[D] \in e[D']$ and $D' \in e[D]$ in $\mathcal{F}$.

As a part of the *Equivalence Theorem* between proof-nets and proof-networks we prove that $B \in e(A)$ in $\mathcal{S}$ if and only if $[B] \in e[A]$ in $\mathcal{F}$.

The strong normalization theorem follows now from the fact that in a family of quasi-structures any cut-reduction reduces the total number of formula-occurrences in the proof-network. As indicated in section (1.1.2), the uniqueness of the normal form is achieved in the sense that different normal derivations are mapped to to the *same set* of quasi-structures.

As a result, we obtain a *three-dimensional* representation of proofs. Different quasi-structures are in parallel planes. The verification of the *multiplicative conditions* (vicious circle and connectedness conditions) is performed separately within each plane. The verification of the *additive conditions* (Box condition) is performed in a plane perpendicular to that of the quasi-structures. The determination of the conclusions of the proof-network and the verification of correctness of the first-order condition are also performed across the planes of the quasi-structures; for instance, we must establish the existence of a bijection between the conclusions of any two quasi-structures in the proof-network.

Such a geometry has a computational interpretation. In the plane of each quasi-structure we operate in a *multiplicative* context, thus there are no restrictions to parallelism (see Girard [1986]). In the orthogonal direction we are concerned with the input-output of each computation as well as of the entire process. Synchronization conditions are introduced at this level.

In this representation we can easily explain in which sense resource-aware logics simplifies the *practical* problem of the representation of the connections of *relevance within a proof*. To the question '*what part of a proof is relevant to a certain statement ?*', we can give a precise answer: '*the empire of the corresponding formula-occurrence*' (in the corresponding proof-structure). A resource-aware logic has the distinctive property that *the relevance connections can be represented on the conclusion of a proof*, simply by projection on the plane of the conclusions.

1.1.4. Two Generalizations.

"Non-Monotonic" Logics? Consider the system of Sequent Calculus for Propositional **MLL** (Multiplicative Linear Logic) and add to this system the structural rule of Mingle. The rule of Mingle allows the combination of two entirely different deductive contexts $\vdash \Gamma, A$ and $\vdash \Delta, B$ into a unique context $\vdash \Gamma, \Delta, A, B$. Now by the $\sqcup$ rule we can infer $\vdash \Gamma, \Delta, A \sqcup B$.

Is this use of $\sqcup$ appropriate with respect to its intended meaning? Has this fragment any useful application? Can this logic be represented by a system of proof-nets as the system without Mingle?

Given deductions $\mathcal{D}_1$ of $\vdash \Gamma, A$ and $\mathcal{D}_2$ of $\vdash \Delta, B$, the proof-structures $\mathcal{S}_1 = \pi(\mathcal{D}_1)$ and $\mathcal{S}_2 = \pi(\mathcal{D}_2)$ still satisfy the vicious circle condition, and do not share any formula-occurrence, thus $A \sqcup B$ violates the connectedness condition, since $e(A) \cap e(B) = \emptyset$. But this property may be regarded as characteristic of the linear $\sqcup$. To avoid any confusion in section (8.1) we use the notation of Direct Logic, with $\wedge$ and $\vee$ instead of $\otimes$ and $\sqcup$ and without propositional constants $\mathbf{1}$ and $\bot$. The Sequent Calculi **DL** and **DL**$^+$ are like those for **MLL**, with the additional structural rules of Mingle and Weakening, respectively.

To the second question we can only answer with a guess. The logic **DL** allows the combination of two entirely distinct propositional contexts and yet at the same time maintains a notation identifying which statements are relevant to which statements. Therefore **DL** could be of use in the treatment of *non-monotonic* forms of reasoning. Or perhaps *the process of separating the contexts*, i.e., the passage from **DL** to **DL**$^+$, can be used in such treatments.

Say that a proof-structure $\mathcal{S}$ for **DL** is a proof-net if in $\mathcal{S}$ no chain is a cycle. This notion provides an answer to the third question.

The difficulty of the Sequentialization Theorem (section 8.8) in this case lies in the fact that here we can no longer apply the argument of Girard's alternative proof. Indeed in the difficult case when all conclusions have the form $A_i \otimes B_i$ for $i \leq n$, the fact that $e(A_j)$ is maximal with respect to $\subsetneq$ does not guarantee the splitting of the structure $\mathcal{S}$ into $e(A_j)$, $e(B_j)$ and $\{A_j \otimes B_j\}$. A more complicated analysis of the graph-theoretic structure is necessary in this context (sections 8.4, 8.5, 8.6).

For the system $\mathbf{DL}^+$ there is a translation $(.)^\ell$ such that given a derivation $\mathcal{D}$ in $\mathbf{DL}^+$ or a proof-structure $\mathcal{S}$ for $\mathbf{DL}^+$, $\mathcal{D}^\ell$ or $\mathcal{S}^\ell$ is a derivation in propositional **MLL** with the $\otimes$ rule or a proof-structure for the same fragment. In order to facilitate a comparison between the '*method of empires*' and the '*method of chains*' we provide also a direct proof of the Sequentialization Theorem for $\mathbf{DL}^+$.

An Interpretation of '!' and '?'. Extensions to the entire system of Linear Logic are possible in the direction of the following remark by Girard:

> "The slicing does not work for !-boxes; this is because the modalities ! and ? are the only nonlinear operations of linear logic. However, if we were working with ! as an infinite tensorization $\bigotimes(\mathbf{1} \sqcap A)$, then it would be possible to slice, but we would get a nondenumerable family of slices! A more reasonable approach would be to make finite developments based on $!A = (\mathbf{1} \sqcap A) \otimes !A$ so that we never go to the ultimate, nondenumerable slicing, but generate it continuously. This idea is of interest because it could serve, without changing anything essential to PN2, for expressing nonterminating processes." (Girard [1987b], Remark 6.3.(ii), page 95)

If we consider first order logic only, then there is certainly no danger that a family of quasi-structures could become a nondenumerable object under the first interpretation. We may define a translation $(.)^\omega$ such that $!A^\omega = \bigotimes(\mathbf{1} \sqcap A^\omega)$ and an operation $\bigotimes_{\mathbf{1} \sqcap A} \mathcal{S}$ as a limit of the operation *Times* $\mathcal{S}\ {}_{1 \sqcap A} \otimes_{1 \sqcap A} \mathcal{S}$. If we consider the representation in terms of a family of quasi-structures, we can see that it satisfies a strong uniformity restriction, since it contains only structures of the form

$$\mathcal{Q}_{i_1}\ {}_{1 \sqcap A} \otimes_{1 \sqcap A} \mathcal{Q}_{i_2}\ {}_{1 \sqcap A} \otimes_{1 \sqcap A} \cdots$$

where $Fam\,\mathcal{S} = \{\mathcal{Q}_1, \mathcal{Q}_2, ...\}$. *Proof Networks* here could be defined as those families $\mathcal{F}$ of structures that satisfy the previous conditions and in addition the *compactness condition*:

No quasi-structure in $\mathcal{F}$ contains an infinite chain without repetition.

We conjecture that a Structure Theorem and a Sequentialization Theorem can be obtained for this notion of proof-net. Then a Strong Normalization Theorem could be obtained for Linear Logic, thus also for Classical Logic, and the ordinal complexity of the proof should be easily measurable from the size of the given data.

However we do not check the details here, for the following reasons. From Gandy [1980] we already know that Strong Normalization for Natural Deduction can be formalized in primitive recursive arithmetic. It is possible that the ordinal of the Strong Normalization Theorem for first order Linear Logic sketched above will also be rather small. On the other hand, we know that from an ordinal proof of Strong Normalization Theorem for *second order* Linear Logic we would obtain an ordinal analysis of second order Peano Arithmetic. Thus it would seem that it is the introduction of second order logic — with full impredicative comprehension axiom — which makes the difference between a relatively small and a huge ordinal. If so, an interesting task would be to develop a theory for (fragments of) second order Linear Logic (see J-Y. Girard, A. Scedrov and P. J. Scott [1990]).

1.1.5. A 'good' Natural Deduction.

Have we obtained at least a good *Natural Deduction system for Linear Logic without exponentials?*

To discuss this question, we propose a notational variant inspired by the *sequent formulation* of Natural Deduction. Suppose we annotate each formula A in a proof-structure $\mathcal{S}$ with

$$(\Gamma)[\Delta] - A,$$

where Γ is the list of *open* doors and Δ is the list of *closed* doors of the empire of A (*fake* and *side* doors in Girard's terminology). In the case of a *first order proof-structure* if a formula $\bigwedge x.A$ is associated with the eigenvariable a, then we add $a <^0 \{b_1 \dots b_n\}$ to the annotations of $\bigwedge x.A$, where $a <^0 b_i$ for all $i \leq n$. In a LISP environment, such annotations may be simply pointers to other parts of

the structure. Notice that in a given proof-structure the empire of a formula is completely determined by its doors.

Considering the properties of the annotations when S is a proof-net, it is not difficult to see that such annotations are the analogue of a list of the open and closed assumptions in ordinary Natural Deduction. For instance, the introduction of a $\otimes$-link may be regarded as analogue to the substitution of a derivation for an open assumption of another derivation, and in this case the annotations of the conclusion depend on the annotations of the premises, as expected.

Similarly, a $\sqcup$-link may be regarded as the analogue of a $\supset$-introduction, and in this case the annotations of the conclusion correspond precisely to the removal of open assumptions in the introduction rule for implication.

Conversely, the annotations may be used to verify whether a proof-structure is a proof-net.

Is this *good enough*? This is open to debate and to experimentation. We may accept the following hint:

> " 'A mathematical proof must be perspicuous'. Only a structure whose reproduction is an easy task is called a 'proof'. It must be possible to decide with certainty whether we really have the same proof twice over, or not. The proof must be a configuration whose exact reproduction must be certain.
>
> ... I want to say: if you have a proof-pattern that cannot be taken in, and by a change of notation you turn it into one that can, then you are producing a proof, where there was none before." (Wittgenstein, *Remarks on the Foundation of Mathematics*, Part III, 1,2.)

Next Wittgenstein asks whether a computation of a Russellian proposition $a+b=c$, where a, b, c are integers represented by sequences of strokes, could be regarded as a proof, in case a and b are large; the answer is no.[3] To the question at the

[3] Undoubtedly aware that such a statement could cause irrelevant discussions, Wittgenstein adds: 'In philosophy it is always good to put a *question* instead of an answer to a question. For an answer to a philosophical question may easily be unfair; disposing of it by means of another question is not. Then should I put a *question* here, for example, instead of the answer that the arithmetical proposition cannot be proved by Russell's method?'

beginning of this section we may therefore simply answer with another question: *Is the sequent formulation of Natural Deduction a good formalization?*

1.2. Realizations and Herbrand's Theorem.

Characteristic of *constructivist* trends is the quest for a *fundamental theory* that encompasses most known forms of sound reasoning and exhibits the rules of certain inference as well as those of effective computation. There may be different views of the progress achieved and the upsets encountered in the pursuit of constructivism, but some facts are established.

Conclusive evidence has been provided that most contemporary mathematics can in principle be reconstructed within the limits of constructive principles: formal techniques have been created that exhibit the constructive content of reasoning, and systematic investigations have been completed on the conditions of their applications. Constructivist proof theory has provided techniques and problems for the classification of functions according to their computational complexity. We have methods to obtain *constructive realizations* of existential statements; namely, given a proof of $\exists x.\varphi(x)$, we can constructively produce a value t satisfying the predicate φ (the *Existential Property*) or, given a proof of $\forall x.\exists y.\varphi(x,y)$, a constructive function f such that $\forall x.\varphi(x, f(x))$. Furthermore, we know that this construction is possible for proofs within intuitionistic logic and that in certain circumstances similar techniques can also be used for classical proofs.

However it cannot be denied that, once Hilbert's 'finitist' approach has been shown inadequate, it becomes difficult to delimit the boundaries of such a fundamental theory, or more precisely, to determine whether or not some other dichotomy should replace Hilbert's opposition *finitary* - *infinitary*, in order to discriminate between the *real* and the *ideal* statements of mathematics. The alternative is to understand constructivism as leading to general methods of providing constructive interpretations of given results, a hermeneutical rather than a fundamental theory.

Part of the problem is that on the one hand a foundational theory of constructive mathematics may include some fragment of the analytical hierarchy as constructive, and on the other hand a theory of feasible computation may want a much narrower definition of constructivity. For practical computation, subclasses of the primitive recursive functions are the most interesting categories. In consequence,

a variety of hierarchies have been introduced and constructive complexity forms a continuous slope along which it is hard to detect the fault line of any fundamental distinction.

In addition, it has been disappointing to discover that constructivist methods that work so successfully *in principle* are not expected to be good candidates for the solution of interesting mathematical problems *in practice*, say, in number theory or in combinatorics — at least not without additional extensive work on the practical conditions of their application. Indeed, one would expect that a fundamental theory would naturally produce applications to different areas of knowledge. Of course, one could say that Kepler's laws did not immediately produce space stations. But these considerations can only encourage us to keep concrete possibilities and practical obstacles in mind while designing formal languages and systems that are intended as contributions to the representation and understanding of mathematical reasoning.

Finally, some of the techniques available to the intuitionistic mathematician are also available to the classical one. Given a proof of $\exists x.\varphi(x)$ in classical predicate calculus, one may apply Herbrand's theorem instead of the intuitionistic existential property and obtain a finite list of values $t_1, \ldots, t_n$ such that $\varphi(t_1) \vee \ldots \vee \varphi(t_n)$. Is a list of values much worse than a single value? If we have a variety of applications in mind, is it worse at all?

Similarly, given a proof of any statement A in Heyting arithmetic, Gödel's *Dialectica Interpretation* produces functionals of higher type that realize the essentially existential quantifiers of A. On the other hand, given a proof of A in Peano Arithmetic, Kreisel's *No Counterexample Interpretation* (NCI) yields a realization by α-recursive functionals of *type 2*, for $\alpha < \epsilon_0$ and, in addition, may save us from introducing inessential dependencies that are required by the Dialectica translation.

Practical applications of these methods heavily depend on the context. If the induction formula is of high quantificational complexity, the definition of the interpretation may become tricky. On the other hand, if it happens that we can restrict ourselves to cuts and induction formulas of limited quantificational complexity, say Π^0_2 cuts and induction formulas, then the interpretation is entirely straightforward. This remains true if the induction rule depends on quantificationally complex assumptions, although in this case the NCI yields realizations that may depend on new function parameters (see the next section). Again, is a parametric result much

worse than a unique result? Can we exploit the presence of parameters to obtain solutions to a family of problems, rather to a single one?

1.2.1. An Application to Ramsey Theory.

We use the term *unwinding* for the extraction of information implicit in a proof, through transformations of either informal arguments or of formalized derivations. The following type of application is of interest to us. Suppose we have a proof of a statement of the form $\forall x.\exists y.\phi(x,y)$, expressing the fact that a certain set H is unbounded. Then the above techniques allow us to define a function(al) F such that

$$\forall x.\phi(x, F(x)) \tag{1}$$

and at the same time provide information about its complexity.

Fruitful use of this method in practice depends on the conditions of the proof. In the simplest and best-known case ϕ is quantifier-free, and (1) is (inductively) derived from universal axioms. What happens in the case of more complex axioms? Consider a formal deduction of a sentence of the form (1) from assumptions of the form

$$\forall u.\exists v.\forall z.\theta(u,v,z) \tag{2}$$

and suppose that (2) contains the definition of the set H. The NCI functional interpretation of (2) $\Rightarrow$ (1) yields functionals U, Z and F, where U and Z are functionals of $\mathbf{v}$, x, such that

$$\forall \mathbf{v} x.\left[\theta(U, \mathbf{v}[U], Z) \supset \phi(x, F(\mathbf{v}, x))\right]$$

The (set of) parameter(s) $\mathbf{v}$ may be crucial in determining the set H' which is defined by $\theta(U, \mathbf{v}[U], Z)$. If we let $\mathbf{v}$ satisfy Skolem axioms of the form $\forall u.\exists v.\forall z.\theta(u,v,z) \supset \forall uz.\theta(u, \mathbf{v}[u], z)$, then H is H'. However the choice of different parameters gives us the opportunity to search for *interesting* sets H'.

In the second part of the dissertation we analyze the Infinite Ramsey Theorem (see Graham, et al. [1980]) and present applications that are relevant to some

finite versions of Ramsey's Theorem.[4] Let $[\mathrm{N}]^k$ be the set of all k-element subsets of the natural numbers N. Given any c-coloring $\chi : [\mathrm{N}]^k \to c$, a subset A of N is *χ-homogeneous* if all k-element subsets of A are monochromatic.

Infinite Ramsey Theorem. (IRT) *For every c, k and any c-coloring $\chi : [\mathrm{N}]^k \to c$, there is an infinite χ-homogeneous set H.*

Let $[n_1, n_2]^k$ be the set of k-element subsets of $\{n_1, \ldots, n_2 - 1\}$, $[n]^k = [0, n]^k$. Say that A is *large* if $|A| > min(A)$.

Finite Ramsey Theorem. (FRT) *For every c, k, l there is a number $n = R(c, k, l)$ such that for any c-coloring $\chi : [n]^k \to c$, there exists a χ-homogeneous subset A of $[n]$, with $|A| = l$.*

Ramsey–Paris–Harrington Theorem. (RPH) *For every c, k and n_1, there exists $n_2 = LR(c, k, n_1)$ such that for every coloring $\chi : [n_1, n_2]^k \to c$ there is a large χ-homogeneous set.*

We consider the case of $c = k = 2$. The Infinite Ramsey Theorem can be formally proved in a system of second order arithmetic. Since k is fixed, in this thesis we use instead first order Peano Arithmetic, with additional function and predicate symbols — we need a binary function symbol for χ and a unary function symbol for an auxiliary χ^{stab}; a unary predicate for H, a unary predicate for an auxiliary set H_1 and a binary predicate for the sequence of sets $S(n)$, where χ^{stab}, H, H_1, $S(n)$ are defined in Section (12) below. Given an interpretation $\mathcal{I}$ for the extended language, $H_{\mathcal{I}}$ (in the standard model) is the desired $\chi_{\mathcal{I}}$-homogeneous set, assuming that the defining properties of χ^{stab}, H, H_1 and $S(n)$ are satisfied in $\mathcal{I}$.

The proof can be divided into two parts: the first part shows that the set H is unbounded and the second that H is χ-homogeneous. Part one has the form $(2) \Longrightarrow (1)$, where (1) expresses that H is unbounded, and (2) contains the formal definition of H and of the auxiliary sets, together with the assumption that χ is a 2-coloring.

The main result is the following. We give a formula expressing restrictions for a "meaningful" choice of the parameters **v**, called the *Non-Triviality Condition* (NTC). This must be satisfied either globally, for all n, or locally, over the segment

[4] Consideration of this example was suggested in Kreisel [1977].

$[x, y]$, in which case we write NTC$[x, y]$. In the statement of the main theorem $F^{(l)}(\chi, \mathbf{v}, n) = F(\chi, \mathbf{v}, F^{(l-1)}(\chi, \mathbf{v}, n))$ is the l-th iteration of F.

Parametrized Ramsey Theorem (PRT): *There is a functional F, primitive recursive*[5] *in χ, $\mathbf{v}$, S_F and H_1, such that for every coloring $\chi : [\mathrm{N}]^2 \to [2]$ and every choice of $\mathbf{v}$,*

(i) *if $\mathbf{v}$ satisfies the* (NTC) *and $p = F^{(l)}(\chi, \mathbf{v}, 1)$, then $[p]$ contains a χ-homogeneous set of cardinality l;*

(ii) *if $\mathbf{v}$ satisfies the* (NTC)$[x, y]$ *and $p_0, \ldots p_l \in [x, y]$, where $p_i = F^{(i)}(\chi, \mathbf{v}, p_0)$, then $[x, y]$ contains a χ-homogeneous set of cardinality l.*

Moreover, for all χ and l, there exists $\mathbf{v}$ satisfying the NTC*; in particular:*

(1) *for any fixed χ, some $\mathbf{v}$ satisfies the* NTC *for all l;*

(2) *for any fixed l, there exist p and $\mathbf{v}$ satisfying* NTC$[0, p]$ *for all χ;*

(3) *for any fixed n_1, there exist n_2 and $\mathbf{v}$ satisfying* NTC$[n_1, n_2]$ *for all χ, with $l = F(\chi, \mathbf{v}, n_1)$.*

Corollary 1: *The Parametrized Ramsey Theorem and Compactness*[6] *imply the Infinite Ramsey Theorem* for $c = k = 2$.

Corollary 2: *The Parametrized Ramsey Theorem implies the Finite Ramsey Theorem* for $c = k = 2$.

Corollary 3: *The Parametrized Ramsey Theorem implies the Erdős-Mills version of the Ramsey-Paris-Harrington Theorem* for $c = k = 2$.

The PRT can be regarded as a generalization of *both* the Infinite and the Finite Ramsey Theorem as shown by the corollaries.[7] This theorem provides a precise mathematical content to the following phenomenological remark. In the proofs of the Ramsey Theorems under consideration, two components can be distinguished:

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[5] In fact, F is in $\mathcal{E}_2(\chi, S_F, H_1, \mathbf{v})$, the third class in the Grzegorczyk hierarchy relativized to χ, $\mathbf{v}$, S_F and H_1.

[6] Since we consider only countable colorings, when we speak of "Compactness" we mean an application of (Weak) König's Lemma.

[7] Remember that we cannot derive the Infinite Ramsey theorem from the Finite version by König's Lemma (at least not in a conservative extension of PA, since the IRT implies the RPH Theorem).

(i) the common frame of the infinite *and* finite versions of the theorem is the structure of their recursive definitions and inductive arguments;[8]

(ii) the properties of the set H — being χ-homogeneous, or of a certain cardinality, or large — depend on the choice of certain set and function parameters, different parameters yielding different versions of the theorem.

Component (i) is represented by the functional $\lambda\chi\ \mathbf{v}.F$; the choices in (ii) are represented by the parameters S and $\mathbf{v}$. The NTC provides constraints for the choices in (ii) to be meaningful. The computational complexity of the proof is determined by the choices in (ii) and by the verification of the NTC, while the computational complexity of F itself is very low.

Although we have not checked it in detail, it seems clear that one can prove the PRT for arbitrary exponents k, following the pattern of the proof given below for $k = 2$.

1.2.2. Use of Direct Logic in Unwinding.

For our unwinding we formalize the proof of the *Infinite Ramsey Theorem using* the notation of Direct Logic for some parts of the proof. Next we apply Herbrand's Theorem to the parts in question and the NCI to the rest.

First notice that it would be foolish to use the translation $(.)^+$ from Classical Logic into Linear Logic. We do not want to prove formulas in which every atom P is replaced by $?!P$; it would be of no interest to study proofs of Ramsey Theorem with *arbitrarily* complex cuts and to compare them with the results of the Cut Elimination procedure. Fortunately, we will succeed in considering only Π^0_2 cuts and induction formulas. On the other hand, using Direct Logic as much as possible in the proof simplifies the procedure, due to the following facts.

Let $\mathcal{L}$ be a first order language. Let S be any sequent in $\mathcal{L}$ and let $\mathcal{L}'$ be an extension of $\mathcal{L}$ containing Herbrand functions for the essentially universal quantifiers of S. We use the boldface $\mathbf{y}$ for the Herbrand function as a formal symbol of the extended language $\mathcal{L}'$, where $\mathbf{y}$ is associated with an essentially universal quantifier $\forall y$ of S. By Herbrand's Theorem, in $\mathcal{L}'$ there are terms $\vec{\mathbf{t}}$ and a quantifier-free

[8] Thus an alternative proof of the Finite Ramsey theorem by double induction (see Graham et al.[1980], p.3) does not belong to this family of proofs.

sequent $\mathcal{E}_{\vec{t}}(S)$ (*the Herbrand expansion of* S) such that S has a normal derivation in Classical Logic if and only if $\mathcal{E}_{\vec{t}}(S)$ has one.

If $\mathcal{E}_{\vec{t}}(S)$ is obtained from S by replacing the essentially universal variables $\vec{y}$ by Herbrand functions $\vec{\mathbf{y}}$ and the essentially existential variables $\vec{x}$ with $\vec{\mathbf{t}}$, then we write S_H for $\mathcal{E}_{\vec{t}}(S)$. Herbrand's Theorem has this particularly simple form in Direct and Multiplicative Linear Logic.

Herbrand's Theorem: *Let* $\phi : \exists x_1.\forall y_1.\ldots\exists x_n.\forall y_n.\psi$ *be in prenex form.*[9] *There are terms* $\mathbf{t}_1,\ldots,\mathbf{t}_n \in \mathcal{L}'$ *such that* ϕ *is provable in First Order Direct Logic if and only if* $\phi_H : \psi(\mathbf{t}_1,\mathbf{y}_1[\mathbf{t}_1],\ldots,\mathbf{t}_n,\mathbf{y}_n[\mathbf{t}_1,\ldots,\mathbf{t}_n])$ *is provable in Propositional Direct Logic.*[10]

The following fact is practically useful:

Lemma. *Let* $\mathcal{D}$ *be a derivation of* S *in first order* MLL *in which the only application of Cut is the last inference and such that the Cut-formula is* Π^0_2. *Let* $S_H(\vec{\mathbf{s}})$ *be the Herbrand expansion of the conclusion, defined by induction on* $\mathcal{D}$. *If* $\mathcal{D}^*$ *is the result of the Cut-elimination procedure and* $S_H(\vec{\mathbf{t}})$ *is the Herbrand expansion of the conclusion defined by induction on* $\mathcal{D}^*$, *then* $\vec{\mathbf{s}} = \vec{\mathbf{t}}$.[11]

1.3. Conclusion.

The technical work in this thesis thus aims at two different problems (1) the search for a new Natural Deduction system enjoying the uniqueness of the normal form, and (2) the extraction of implicit information from a given mathematical proof by the application of functional interpretations. In both cases it has been useful to consider Linear Fragments of Classical Logic. We believe that work in this area is also producing more sophisticated tools to analyze deductive relevance and irrelevance.

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[9] The restriction is essential, as shown by the counterexample $\forall x.(\phi(x) \wedge \psi) \not\vdash (\forall y.\phi(y)) \wedge \psi$, while $\phi(\mathbf{y}) \wedge \psi \vdash \phi(\mathbf{y}) \wedge \psi$, as mentioned in section (1.1.1).

[10] The same property holds if conditional terms `if...then...else...` can be expressed in $\mathcal{L}$. In this case we may call the terms $\vec{\mathbf{t}} \in \mathcal{L}'$ *functionals of predicate logic.*

[11] This is not true in general for Direct Logic: if the Cut-formula is introduced by Weakening, say, in the derivation of the left premise, then in $\mathcal{D}^*$ all the side formulas of the right premise are introduced by Weakening, thus some terms in $\vec{\mathbf{t}}$ may be simpler than some terms in $\vec{\mathbf{s}}$.

The techniques presented here are not recommended for *all* uses. For instance, we introduced *Families of Quasi-Structures for Linear Logic* in order to attack problem (1), but preferred the notation of Direct Logic for problem (2). In fact we do not expect that progress in the two directions will depend on the same technical improvements. Perhaps the following question is relevant here:

Are we ready at the present stage of research to present a unified theory of proofs that accounts for all major functions of efficient deductive reasoning through the tools of a formal representation?

Wittgenstein's remark quoted in section (1.1.5) applies to formal and mechanical representations of proofs insofar as they intend to maintain the features of informal reasoning that make it suitable to *convey evidence*. But the same remark applies *mutatis mutandis* to formal representations insofar as they intend to exhibit the *computational* content of a proof.

Unfortunately, there seems to be no guarantee that what is efficient to convey evidence will also be efficient to represent a computation. In the introduction to his thesis, also focusing on computational applications of Proof Theory, C.Goad says:

> "It is necessary to keep computational considerations explicitly in mind when constructing proofs which are intended as descriptions of computation. The best proof of a formula $\forall x.\exists y.\varphi(x,y)$ according to such standard criteria as brevity, elegance or comprehensibility, will often embody a very bad algorithm. Conversely, a proof of $\forall x.\exists y.\varphi(x,y)$ which formalizes a good algorithm will generally constitute a rather unnatural way of establishing the simple truth of the formula."(Goad [1980])

However, it is not impossible that improvements in understanding of informal reasoning may help us make our formal languages more concise, elegant or comprehensible and *at the same time* more efficient as tools for computation. For instance, it seems that the study of *resource-aware* logics improves our understanding of the notion of relevance within a proof, and at the same time simplifies the verification of functionals defined in the process of unwinding. Similarly, one may hope that the production of formalisms that are not only ecological but also *efficient in accounting of the necessary resources* would be fruitful for computation and instructive for our understanding of informal mathematical reasoning.

PART I

2. Language.

The language $\mathcal{L}_L$ for Girard's Commutative and Noncommutative Linear Logic is described (see Girard [1987b]) and some notational conventions are established.

$\mathcal{L}_L$ is a first order language containing symbols for n-ary predicates and functions, for $n \geq 0$, the propositional constants $\top$, $\mathbf{1}$, $\bot$ and $\mathbf{0}$, the unary propositional operations $()^{\bot}$ (*linear negation*), ! (*of course!*) and ? (*why not?*), the binary connectives $\otimes$ (*times*), $\sqcup$ (*par*), $\sqcap$ (*with*) and $\oplus$ (*plus*), and the first order quantifiers $\bigwedge$ (*all*) and $\bigvee$ (*some*).

The notion of a term and a formula in the language $\mathcal{L}_L$ is defined in the usual manner.

Definition. Any variable is a term. If $t_1, \ldots, t_n$ is a list of terms and f is an n-ary function symbol, then $f(t_1, \ldots, t_n)$ is a term. If $t_1, \ldots, t_n$ is a list of terms and P is an n-ary predicate symbol, then $P(t_1, \ldots, t_n)$ is an atomic formula. The propositional constants $\top$, $\mathbf{1}$, $\bot$ and $\mathbf{0}$ are atomic formulas. If A is a formula, then $A^{\bot}$ is a formula. If A is a formula, then $!A$ and $?A$ are formulas. If A and B are formulas, then so are $A \otimes B$, $A \sqcup B$, $A \sqcap B$ and $A \otimes B$. If A is a formula and x is a variable, then $\bigwedge x.A$ and $\bigvee x.A$ are formulas.

For any formula A and a list $(t_1, \ldots, t_n)$ of terms, we use the notation $A(t_1, \ldots, t_n)$ for A with all the free occurrences of the variable x_i replaced by t_i for $i = 1, 2, \ldots, n$.

Remark 1. In *Non-Commutative Linear Logic* the following are valid schemata.

$$\begin{gathered}
\mathbf{1}^{\bot} = \bot, \quad \bot^{\bot} = \mathbf{1}, \quad \top^{\bot} = \mathbf{0}, \quad \mathbf{0}^{\bot} = \top, \\
A^{\bot\bot} = A, \\
(A \otimes B)^{\bot} = B^{\bot} \sqcup A^{\bot}, \quad (A \sqcup B)^{\bot} = B^{\bot} \otimes A^{\bot}, \\
(A \sqcap B)^{\bot} = A^{\bot} \oplus B^{\bot}, \quad (A \oplus B)^{\bot} = A^{\bot} \sqcap B^{\bot}, \\
(!A)^{\bot} = ?(A^{\bot}), \quad (?A)^{\bot} = !(A^{\bot}), \\
(\bigwedge x.A)^{\bot} = \bigvee x.(A^{\bot}), \quad (\bigvee x.A)^{\bot} = \bigwedge x.(A^{\bot}).
\end{gathered}$$

Using these laws we can reduce the scope of linear negation. We therefore adopt the following convention:

Linear Negation will be applied only to atomic formulas.

The following *Linear implications* can be defined:

$$A \multimap B =_{df} A^{\perp} \sqcup B \quad \text{and} \quad A \circ\!- B =_{df} A \sqcup B^{\perp}.$$

Remark 2. In *Commutative Linear Logic* the schemata listed in Remark 1 are valid and, in addition, we have

$$A \otimes B = B \otimes A, \qquad A \sqcup B = B \sqcup A \qquad A \multimap B = B \circ\!- A.$$

Therefore, the following laws will be applied to reduce the scope of linear negation in Commutative Linear Logic:

$$(A \otimes B)^{\perp} = A^{\perp} \sqcup B^{\perp}, \quad (A \sqcup B)^{\perp} = A^{\perp} \otimes B^{\perp}.$$

2.1. The Sequent Calculus LL.

We describe the sequent calculus **LL** presented in Girard [1987b], with only minimal variations, and we list the subsystems of **LL** of interest to us.

We work in the language $\mathcal{L}_L$. A sequent S, written as $\vdash \Gamma$, is a finite sequence of formulas. We use Greek capital letters for finite sequences of formulas. As usual $\vdash \Gamma, A$ stands for $\vdash \Gamma * \langle A \rangle$, where $*$ is the concatenation operation. If Γ is $A_1, \ldots, A_n$, then Γ is interpreted as $A_1 \sqcup \ldots \sqcup A_n$, using *multiplicative* disjunction.

Remark. According to the standard usage, the symbol "$\vdash$" is here a purely formal notation, albeit a suggestive one, that characterizes $\vdash \Gamma$ as an expression of the sequent calculus.

Logical axioms are sequents $\vdash P^{\perp}, P$ (where P is atomic).

Non-logical axioms (or nonlogical initial sequents) are arbitrary sequents $\vdash \Gamma$.

The only *structural rule* is Exchange.

Exchange

Commutative	**Non Commutative**
$\dfrac{\vdash \Gamma, A, B, \Delta}{\vdash \Gamma, B, A, \Delta}$	$\dfrac{\vdash \Gamma, A}{\vdash A, \Gamma.}$

Remark. By iteration of the Noncommutative or Commutative Exchange Rule we obtain permutations of the given sequent that are cyclic or arbitrary, respectively. We do not obtain an interesting logic if Exchange is dropped entirely. In fact, we insist that the rule Cut must behave exactly like the *times* rule, for reasons that will become clear later. Given such a requirement on Cut, the absence of any Exchange rule induces unacceptable restrictions to the type of formulas which Cut can be applied to.

Multiplicative rules.

1-Axiom

$$\vdash \mathbf{1}$$

$\bot$ - Rule

$$\frac{\vdash \Gamma}{\vdash \Gamma, \bot}$$

Conjunction (times)

$$\frac{\vdash \Gamma, A \qquad \vdash B, \Delta}{\vdash \Gamma, A \otimes B, \Delta}$$

Disjunction (par)

$$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \sqcup B}$$

Additive rules.

$\top$-Axiom

$$\vdash \top, \Gamma$$

(no rule for **0**)

Conjunction (with)

$$\frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \sqcap B}$$

Disjunction (plus)

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B} \qquad \frac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B}$$

Quantification rules (first order).

Universal Quantification

$$\frac{\vdash \Gamma, A(a)}{\vdash \Gamma, \bigwedge x.A(x)},$$

where the variable a does not occur in Γ.

Existential Quantification

$$\frac{\vdash \Gamma, A(t)}{\vdash \Gamma, \bigvee x.A(x)}$$

Exponentiation rules.

Dereliction

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}$$

! (of course)

$$\frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}$$

where $?\Gamma$ is a sequence $C_1, \ldots, C_m$ and
for all $i \leq n$ C_i is either $\perp$ or of the form $?B_i$.

Weakening

$$\frac{\vdash \Gamma, \perp}{\vdash \Gamma, ?A}$$

Contraction

$$\frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}$$

In addition, we have the **Cut** *rule:*

$$\underline{\vdash \Gamma, A \vdash A^{\perp}, \Delta}$$

According to the standard terminology, the formula A, occurring positively in the left upper sequent and negatively in the right upper sequent is called *Cut formula.*

Derivations are inductively defined as usual. Also the notion of *branch* in a derivation is as usual. In each of the above rules, the indicated formulas in the upper sequent(s) are the *active* formulas and the indicated formula in the lower sequent the *principal* formula. All the other formulas in the indicated sequences are *side* formulas. The active formula(s) are the *immediate ancestors* of the principal formula and the relation "*... is an ancestor of ...*" is the transitive closure of "*... is an immediate ancestor of ...*".

Remark. In the inferential step from $\vdash \Gamma, A$ to $\vdash \Gamma, \bigwedge x.A$ we must keep the occurrences of x in the sequent and in the derivation under control. Following the standard practice, we work with an infinite additional set of *eigenvariables* (or

parameters) a, b,... and we assume that for every principal formula $\bigwedge x.A$, $\bigwedge y.B$ in a derivation, *the eigenvariable a of $\bigwedge x.A$ is different from the eigenvariable b of $\bigwedge y.B$.* In particular, every derivation enjoys the *pure parameter property: the eigenvariable a associated with $\bigwedge x.A$ occurs only above the inference*

$$\frac{\vdash \Gamma, A(a)}{\vdash \Gamma, \bigwedge x.A.}$$

Definition. The system of sequent calculus **LL** for First Order Commutative Linear Logic is based on the language $\mathcal{L}_L$ and contains Logical Axioms, the Commutative Exchange Rule and the Multiplicative, Additive, Quantification and Exponentiation Rules. We consider the following subsystems:

(*i*) propositional [and first order] $\mathbf{MLL}^-$ is the multiplicative fragment without $\mathbf{1}$ and $\perp$ and contains only Logical Axioms, the Commutative Exchange Rule, the rules for *times* and *par* [and the quantification rules].

(*ii*) propositional [and first order] **MLL** is the multiplicative fragment, and contains the the axiom for $\mathbf{1}$ and the rule for $\perp$ in addition to the axioms and rules of propositional [and first order] $\mathbf{MLL}^-$.

(*iii*) propositional [and first order] **MALL** is the full multiplicative and additive fragment, and contains axioms and rules of propositional [and first order] **MLL** plus the axiom for $\top$ and the rules for *with*, *plus*.

(*iv*) propositional and first order $\mathbf{NCMLL}^-$, **NCMLL** and **NCMALL** are the sequent calculi for *Non Commutative* Linear Logic are like the corresponding commutative systems, but allow only the Non Commutative Exchange Rule.

2.2. A Note on Semantics.

We give a short description of two semantics for Linear Logic presented in Girard [1987b], for the convenience of the reader.

Proofs in Commutative Linear Logic can be interpreted in the category of *Coherent Spaces* and *Linear Maps.* (see Girard [1987b], pp. 14-17 and 48-60).

Given a set S, a *coherent space* X is a set of subsets of S satisfying the two conditions

(i) if $a \in X$ and $b \subset a$, then $b \in X$;

(ii) if $A \subset X$ and for all $a, b \in A$ we have $a \cup b \in X$, then $\cup A \in X$.

A *linear map* $F : X \to Y$ of coherent spaces is a function satisfying the following four conditions:

(i) if $a \subset b \in X$ then $F(a) \subset F(b)$;

(ii) if $a \cup b \in X$ then $F(a \cap b) = F(a) \cap F(b)$;

(iii) if $a \cup b \in X$ then $F(a \cup b) = F(a) \cup F(b)$;

(iv) $F(\emptyset) = \emptyset$.

The coherent spaces $X \otimes Y$ and $X \sqcup Y$ are variants of the cartesian product of coherent spaces and the coherent spaces $X \sqcap Y$ and $X \oplus Y$ are variants of the direct sum (disjoint union) of coherent spaces. For this reason, $\mathbf{1}$, $\perp$, $\otimes$ and $\sqcup$ are called *multiplicatives* $\top$, $\mathbf{0}$, & and $\oplus$ *additives*.

Other non-standard semantics have been given for Commutative Linear Logic. (see Girard [1987b], Lafont [1988], Martí-Oliet and Meseguer [1989]). We mention only the *Phase Semantics* (Girard [1987b], pp.17-28).

Definition. A *phase space* is a pair $(\mathbf{P}, \perp)$, where $(\mathbf{P}, \cdot, 1)$ is a commutative monoid and $\perp$ is distinguished subset. Given any subset X of $\mathbf{P}$, let $X^{\perp}$ be $\{p \in \mathbf{P} : p \cdot X \subset \perp\}$. A *fact* is a subset F such that $F^{\perp\perp} = F$.

A *topolinear space* $(\mathbf{P}, \perp, \mathbf{F})$ is a phase space $(\mathbf{P}, \perp)$ together with a set $\mathbf{F}$, the set of *closed facts* of $(\mathbf{P}, \perp)$, satisfying the conditions

(i) $\mathbf{F}$ is closed under arbitrary intersections ($\sqcap$);

(ii) $\mathbf{F}$ is closed under finite $\sqcup$, so $\perp \in \mathbf{F}$;

(iii) $\perp \subset F$, for all $F \in \mathbf{F}$;

(iv) $F \sqcup F = F$, for all $F \in \mathbf{F}$.

If $F \in \mathbf{F}$, then $F^{\perp}$ is an *open fact*.

An interpretation $\mathcal{I}$ of $\mathcal{L}_L$ in $(\mathbf{P}, \perp)$ is an assignment of facts to the formulas satisfying the definition below. We assign the subset $\perp$ to the symbol $\perp$ and the subset $\mathbf{P}$ to the symbol $\top$ — it is easy to prove that the subset $\perp$ and $\mathbf{P}$ are facts.

If P is an atomic formula, then $P_{\mathcal{I}}$ is a fact and $(P^{\perp})_{\mathcal{I}} = (P_{\mathcal{I}})^{\perp}$. Moreover

(1) $(A \otimes B)_{\mathcal{I}} = [(A)_{\mathcal{I}} \cdot (B)_{\mathcal{I}}]^{\perp\perp}$;

(2) $(A \sqcup B)_{\mathcal{I}} = [(A)_{\mathcal{I}}^{\perp} \cdot (B)_{\mathcal{I}}^{\perp}]^{\perp}$;

(3) $(A \sqcap B)_{\mathcal{I}} = (A)_{\mathcal{I}} \cap (B)_{\mathcal{I}}$;

(4) $(A \oplus B)_{\mathcal{I}} = [(A)_{\mathcal{I}} \cup (B)_{\mathcal{I}}]^{\perp\perp}$.

Finally in a topolinear space we define

(5) $(?A)_{\mathcal{I}}$ = the smallest closed fact containing $A_{\mathcal{I}}$;

(6) $(!A)_{\mathcal{I}}$ = the largest open fact contained in $A_{\mathcal{I}}$.

A *phase structure* is a phase space $(\mathbf{P}, \perp)$ together with an interpretation $\mathcal{I}$. A formula A is *valid in* a phase structure $((\mathbf{P}, \perp), \mathcal{I})$ if $1 \in A_{\mathcal{I}}$. A *topolinear structure* is a phase space $(\mathbf{P}, \perp, \mathbf{F})$ together with an interpretation $\mathcal{I}$. A formula A is *valid in* $((\mathbf{P}, \perp, \mathbf{F}), \mathcal{I})$ if $1 \in A_{\mathcal{I}}$. A formula A [without exponentials] is *linear tautology* if A is valid in any topolinear structure [phase structure].

The following topolinear space may be regarded as the canonical model for **LL** in Phase Semantics. Let $\mathbf{P}$ be the set of *multisets* of formulas of $\mathcal{L}_L$ (i.e., sequences of formulas, up to their order). $(\mathbf{P}, *, \emptyset)$ is a monoid, where $*$ is concatenation of multisets. We let $\perp$ be the set of multisets Γ such that $\vdash \Gamma$ is provable in the sequent calculus **LL**. Then we can show that $((\mathbf{P}, *), \emptyset, \perp)$ is a phase space. We define a canonical interpretation $\mathcal{I}$ of $\mathcal{L}_L$ by letting

$$A_{\mathcal{I}} =_{df} \{\Gamma : \vdash_{\mathbf{LL}} \Gamma, A\}.$$

Furthermore, we define $\mathbf{F}$ as follows. For all facts $X \subset \mathbf{P}$,

$$X \in \mathbf{F} \text{ if and only if } X = \bigcap_i (?A_i)_{\mathcal{I}}, \text{ for some sequence } A_1, A_2 \ldots.$$

It is easy to prove the consistency and completeness of **LL** with respect to Phase Semantics. In particular, if A is a linear tautology, then $A_{\mathcal{I}}$ is true in the canonical model for any interpretation $\mathcal{I}$. Thus $\emptyset \in A_{\mathcal{I}}$, i.e., $\vdash A$ is provable in **LL**.

3. Proof-Structures for Linear Logic.

Definition. A *proof-structure* $\mathcal{S}$ consists of

(i) a nonempty set of *formula-occurrences*, i.e., of indexed formulas, (or, equivalently, a *multiset* of formulas) together with

(ii) a set of relations, or *links*, between formula-occurrences; if a list of distinct formula-occurrences is in a link, then the list can be divided in two sublists, the *premises* and the *conclusions*. A link is *unary* or *binary* if it has one or two premises, respectively.

(iii) In a proof-structure formula-occurrences and links must satisfy the properties that

(1) every formula-occurrence is the conclusion of one and only one link;

(2) every formula-occurrence is the premise of at most one link.

(iv) a partial function $\varphi : \mathcal{S} \rightharpoonup \mathcal{S}$, called *flagging function.*

Remark. There is another natural representation of proof-structures, describing them as *graphs*, whose nodes are labeled with formulas and whose edges connect either a premise and the conclusion of a link, or all the conclusions of a link. We will describe proof-structures in abstract graph-theoretic terms in section (8.4).

Links in a proof structure can be represented as configurations of the forms indicated below. The formula-occurrence[s] above the line is [are] the *premise*[*s*] and the formula-occurrence[s] below the line is [are] the *conclusion*[*s*] of the link.

Definition. A link (represented by a configuration) of the form

$$\overline{X_1, \ldots, X_n}$$

is called an *axiom link.* Axiom links of the form

$$\overline{P, P^\perp, \perp, \ldots, \perp} \quad \overline{\mathbf{1}, \perp, \ldots, \perp} \quad \overline{\top, X_1, \ldots, X_n}$$

are called *logical axioms*, **1** *axioms* and $\top$ *axioms*, respectively. Axiom links of a different form will be called *non-logical axioms*.

A link (represented by a configuration) of one of the forms

$$\frac{X \quad Y}{X \otimes Y} \quad \frac{X \quad Y}{X \sqcup Y} \quad \frac{X \quad Y}{X \sqcap Y} \quad \frac{X}{X \oplus Y} \quad \frac{Y}{X \oplus Y} \quad \frac{X}{\bigwedge x.X} \quad \frac{X}{\bigvee x.X}$$

is called a $\otimes$-, $\sqcup$-, $\sqcap$-, $\oplus$-, $\bigwedge$-, $\bigvee$-*link*, respectively, in accordance with the outermost logical symbol in the conclusion. A binary link (represented by a configuration) of the form

$$\frac{D' \quad D''}{D}$$

where D, D' and D'' are distinct occurrences of the same formula, is a *Contraction link.* A *Cut link* is a binary link

$$\frac{X \quad X^{\perp}}{\underline{X \otimes X^{\perp}}}$$

which is exactly like a $\otimes$-link, except that *the conclusion* $\underline{X \otimes X^{\perp}}$ *cannot be a premise of any link.* We call a formula-occurrence $\underline{X \otimes X^{\perp}}$ a *ghost formula-occurrence.* A formula-occurrence which is not the premise of any link is a *lowermost formula* of the proof-structure. The lowermost formula-occurrences of a proof-structure are either *conclusions* or *ghost formula-occurrences.* We write **CUT** for the set of all ghost formula-occurrences in a proof-structure. We sometimes write $\mathcal{S}_{\varphi}(\Delta)$ for a proof-structure with conclusion Δ and flagging function φ.

Remark. The representation of the cut link is intentionally ambiguous. In the propositional multiplicative fragment, from the point of view of the geometry of the proof-structure, the ghost occurrence behaves exactly as a real formula-occurrence — differences occur the first order case and in the fragment with the additives. However, when we consider a proof-structure as a representation of a derivation, the *ghost formula-occurrence* $\underline{X \otimes X^{\perp}}$ does not count as a conclusions of the proof-structure. Girard [1987b] uses instad a special symbol CUT, which is not a formula. Whatever convention we choose, the ambiguity of the notation is meaningful and suggestive, as a mark of a local contradiction and a temporary instability (see Girard [1989a], II.8.: "the only dynamical figure of the system is the cut-rule; without cut, there would be no action performed").

Definition. Continuing the definition of proof-structure, we require

(3) A link $\frac{X \quad Y}{X \circ Y}$ is *different* from the link $\frac{Y \quad X}{Y \circ X}$. The axiom link $\overline{X_1, \ldots, X_n}$ is identified with the axiom link $\overline{X_{\sigma(1)}, \ldots, X_{\sigma(n)}}$, where σ is any permutation.

The use of Contraction links in a structure $\mathcal{S}$ is restricted as follows:

(4) The *flagging* function φ associated with $\mathcal{S}$ has the following property. For each contraction link

$$\frac{D' \quad D''}{D}$$

there is a $\sqcap$-link

$$\frac{X_0 \quad X_1}{X_0 \sqcap X_1}$$

such that $\varphi(D') = X_i$ and $\varphi(D'') = X_{1-i}$, for $i = 0, 1$. In this case we say that D' is flagged with X_i and D'' is flagged with X_{1-i}. We also put $\varphi(X_j) = X_j$, for $j = 0, 1$.

Definition. Proof-structures for the following fragments are defined by following restriction:

(i) A proof-structure for propositional commutative $\mathbf{MLL}^-$ contains logical axioms and $\otimes$-, $\sqcup$- and cut links.

(ii) A proof-structure for propositional commutative **MLL** contains logical and **1** axioms and $\otimes$-, $\sqcup$- and cut links.

(iii) A proof-structure for propositional commutative **MALL** in addition the links of **MLL** contains **T** axioms and $\sqcap$-, $\oplus$- and Contraction links.

(iv) Proof-structures for the corresponding first-order fragments contain in addition $\bigwedge$- and $\bigvee$-links.

(v) The above fragments are extended by allowing proof-structures with nonlogical axioms.

The premises of a (non-axiom) link are the *immediate ancestors* of the conclusion in a proof-structure. The relation "... *is an ancestor of* ..." is the transitive closure of "... *is an immediate ancestor of* ...". Symmetrically, we define *immediate descendant* and *descendant.*

Remark. We work with proof-structures containing only finitely many formula-occurrences, but in section (1.1.4) we described an extension of our methods to the

case of infinitary proof-structures. In this case we must introduce a well-foundedness requirement, e.g.,

Every formula-occurrence in any axiom of a proof-structure has only finitely many descendants.

Notation. We write $X \sim X'$ if X and X' are different occurrences of the same formula. We write $X \prec Y$ if X is an ancestor of Y in a proof-structure $\mathcal{S}$ and $X \preceq Y$ if X is an ancestor of Y or X and Y are the same formula-occurrence in $\mathcal{S}$.

3.1. Inductive Proof-Structures for MLL.

A class of proof-structures is defined that naturally corresponds to the class of sequent derivations in **MLL**.

Definition. The class of *inductive proof-structure* for first order **MLL** is the smallest class closed under conditions $(1)', (2) - (5)$ below.

(1) $$\overline{P, P^{\perp}},$$

(1)′ $$\overline{P, P^{\perp}, \perp_1, \ldots, \perp_n} \quad \text{and} \quad \overline{\mathbf{1}, \perp_1, \ldots, \perp_n}$$

are inductive proof-structures, for any P.

(2) If $\mathcal{S}'(A_1, \ldots, A_n, A)$ and $\mathcal{S}''(B, B_1 \ldots, B_m)$ are inductive proof-structures and do not share any formule-occurrence, then $\mathcal{S}$

$$\begin{array}{cccc} & \mathcal{S}' & & \mathcal{S}'' \\ A_1, \ldots, A_n, & \dfrac{A \qquad B}{A \otimes B} & & , B_1 \ldots, B_m \end{array}$$

is an inductive proof-structure. We write also $\mathcal{S}' \;_{A \otimes B}\, \mathcal{S}''$ for $\mathcal{S}$.

(2)′ Same as (2), with a ghost formula $\underline{X \otimes X^{\perp}}$ instead of $A \otimes B$.

(3) If $\mathcal{S}'(A_1, \ldots, A_n, A, B)$ is an inductive proof-structure, then $\mathcal{S}$

$$\begin{array}{cc} & \mathcal{S}' \\ A_1, \ldots, A_n, & \dfrac{A \qquad B}{A \sqcup B} \end{array}$$

is an inductive proof-structure.

(4) If $\mathcal{S}'(A_1, \ldots, A_n, A(a))$ is an inductive proof-structure, where a is an eigenvariable, and a does not occur in $A_1, \ldots, A_n$, but may occur in the ghost formula-occurrences, then $\mathcal{S}$

$$\begin{array}{cc} \multicolumn{2}{c}{\mathcal{S}'} \\ A_1, \ldots, A_n, & \dfrac{A(a)}{\bigwedge x.A(x)} \end{array}$$

is an inductive proof-structure.

(5) If $\mathcal{S}'(A_1, \ldots, A_n, A(t))$ is an inductive proof-structure, where t is any term, then $\mathcal{S}$

$$\begin{array}{cc} \multicolumn{2}{c}{\mathcal{S}'} \\ A_1, \ldots, A_n, & \dfrac{A(t)}{\bigvee x.A(x)} \end{array}$$

is an inductive proof-structure.

Definition We define the class of *inductive proof-structure for*

propositional $\mathbf{MLL}^-$ by using only clauses (1), (2), (3),

propositional **MLL** by using only clauses (1)′, (2), (3),

first order $\mathbf{MLL}^-$ by using only clauses (1), (2) – (5).

The class of *inductive proof-structure with nonlogical axioms* for some subsystem is defined with the addition of the clause

$$(0) \qquad \overline{X_0, \ldots, X_n}$$

is an inductive proof-structure, for any $X_0, \ldots, X_n$.

3.2. Inductive Proof-Structures for MALL.

Definition An *inductive proof-structure* (with nonlogical axioms) in propositional or first order **MALL** is defined as for **MLL** with the addition of the following specifications and of clauses (6) and (7):

- If $\mathcal{S}$ is an axiom, then the flagging function φ is empty.
- In case (2) the flagging function φ of $\mathcal{S}'_{\varphi'}\ {}_A\!\otimes_B \mathcal{S}''_{\varphi''}$ is defined by

$$\varphi(X) = \begin{cases} \varphi'(X) & \text{if } X \in \mathcal{S}'; \\ \varphi''(X) & \text{othervise.} \end{cases}$$

• In cases (3) – (5) the flagging function of $\mathcal{S}$ is the same as the flagging function of $\mathcal{S}'$.

(1)″
$$\overline{\top, X_1, \ldots, X_n}$$

is an inductive proof-structure, for any $X_1, \ldots, X_n$.

(6) If $\mathcal{S}'(\Lambda, A)$ and $\mathcal{S}'(\Lambda, B)$ is an inductive proof-structure with flagging function φ, then

$$\begin{array}{cc} \mathcal{S}' & \\ \Lambda, & \dfrac{A}{A \oplus B} \end{array} \quad \text{and} \quad \begin{array}{cc} \mathcal{S}' & \\ \Lambda, & \dfrac{B}{A \oplus B} \end{array}$$

are a inductive proof-structures $\mathcal{S}$ with flagging function φ.

(7) If $\mathcal{S}'(\Lambda', A)$ and $\mathcal{S}''(\Lambda'', B)$ are inductive proof-structures that do not share any formula-occurrence, and Λ, Λ' and Λ'' are different occurrences of the same sequence D_1, D_2,..., D_n, let $\mathcal{S}$ be $\mathcal{S}' \cup \mathcal{S}'' \cup \Lambda \cup \{A \sqcap B\}$ with the addition of the following links:

$$\frac{D_1' \quad D_1''}{D_1} \cdots \frac{D_n' \quad D_n''}{D_n} \qquad \frac{A \qquad B}{A \sqcap B}$$

Then $\mathcal{S}$ is an inductive proof-structure. Let φ_1 and φ_2 be the flagging functions of $\mathcal{S}'$, $\mathcal{S}''$. Define φ by

$$\varphi(X) = \begin{cases} \varphi_1(X), & \text{if } X \in \mathcal{S}'; \\ \varphi_2(X), & \text{if } X \in \mathcal{S}''; \\ A, & \text{if } X \text{ is } D_i', \text{ for } i \leq n, \text{ or if } X \text{ is } A; \\ B, & \text{if } X \text{ is } D_i'' \text{ for } i \leq n, \text{ or if } X \text{ is } B. \end{cases}$$

3.3. Embeddings and Substructures.

Definition. (i) Let $m : \mathcal{S}_\varphi \to \mathcal{S}'_{\varphi'}$ be any map of proof-structures, i.e., a map of $\mathcal{S}$ and $\mathcal{S}'$ regarded as sets (of formula occurrences). m is said to *preserve the links*

$$\frac{A}{B} \qquad \frac{A \quad B}{C} \qquad \overline{P_1, \ldots, P_n}$$

if we have

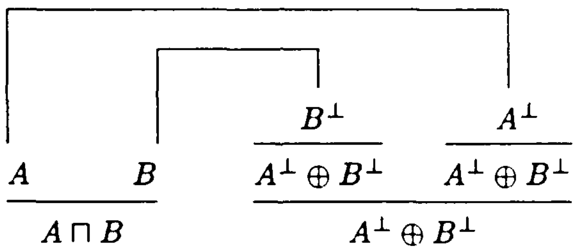
$$\frac{m(A)}{m(B)} \qquad \frac{m(A) \qquad m(B)}{m(C)} \qquad \overline{m(P_1), \ldots, m(P_n), Q_1, \ldots, Q_m}$$

and for every $X, Y \in \mathcal{S}_\varphi$, $\varphi(X) = Y$ if and only if $\varphi'(m(X)) = m(Y)$. (Thus the property of preserving links includes the condition of preserving the flagging function.) We say the m *strongly preserves the axiom link* $\overline{P_1, \ldots, P_n}$ *of* $\mathcal{S}_\varphi$ *if* $\overline{m(P_1), \ldots, m(P_n)}$ *is an axiom of* $\mathcal{S}'_{\varphi'}$. Such a condition guarantees that non-logical axioms of $\mathcal{S}_\varphi$ are sent to non-logical axioms of $\mathcal{S}'_{\varphi'}$.

(ii) An injective map $m : \mathcal{S} \to \mathcal{S}'$ preserving links is called an *embedding* if for each $X \in \mathcal{S}$, $X \sim m(X)$, i.e., if X and $m(X)$ are different occurrences of the same formula.

Definition. (i) Let $\mathcal{S}_\varphi$ be a proof-structure. A proof-structure $\mathcal{S}'_{\varphi'}$ is a *substructure* of $\mathcal{S}_\varphi$ if $\mathcal{S}'_{\varphi'} \subset \mathcal{S}_\varphi$ as sets of formula-occurrences, and the identity map $\iota : \mathcal{S}'_{\varphi'} \to \mathcal{S}_\varphi$ strongly preserves axiom-links and preserves links and the flagging function.

Example. Consider the following proof-structure $\mathcal{S}_\varphi$, where φ maps the left premise of the Contraction link to B and the right premise to A.



Then $\mathcal{S}' = \{A, A^\perp\}$ with the axiom link $\overline{A, A^\perp}$ is a substructure of $\mathcal{S}_\varphi$ and so is $\mathcal{S}'' = \{A, A^\perp, B, B^\perp\}$ with the axiom links $\overline{A, A^\perp}$ and $\overline{B, B^\perp}$. But $\mathcal{S}'''$ = the set of conclusions of $\mathcal{S}'_\varphi$, with axiom link $\overline{A \sqcap B, A^\perp \oplus B^\perp}$ is not a substructure of $\mathcal{S}_\varphi$.

Definition. (continued) (ii) Let $\mathcal{S}_\varphi$ be a proof-structure and let $\mathcal{S}'_{\varphi'}$ be a substructure of $\mathcal{S}_\varphi$ with conclusion Δ. We define $\overline{\mathcal{S}}_{\overline{\varphi}}$ as follows. The formula occurrences in $\overline{\mathcal{S}}_{\overline{\varphi}}$ are those in

$$\Delta \cup (\mathcal{S}_\varphi \setminus \mathcal{S}'_{\varphi'});$$

the links of $\overline{\mathcal{S}}_{\overline{\varphi}}$ are all the links of $\mathcal{S}_\varphi$ defined on such a set, with the addition of the axiom link $\overline{\Delta}$, where Δ is the set of conclusions of $\mathcal{S}'_{\varphi'}$; the flagging function $\overline{\varphi}$ of $\overline{\mathcal{S}}_{\overline{\varphi}}$ is the restriction of φ to the set in question, when defined. Then $\overline{\mathcal{S}}_{\overline{\varphi}}$ is called the *substructure of* $\mathcal{S}_\varphi$ *complementary to* $\mathcal{S}'_{\varphi'}$.

Remark. Notice the abuse of terminology: the substructure of $\mathcal{S}_\varphi$ complementary to $\mathcal{S}'_{\varphi'}$ fails to satisfy the condition for a substructure of $\mathcal{S}_\varphi$, since the axiom $\overline{\Delta}$ is not preserved.

Fact. *If $\overline{\mathcal{S}}_{\overline{\varphi}}$ is the substructure of $\mathcal{S}_\varphi$ complementary to $\mathcal{S}'_{\varphi'}(\Delta)$, then the result of replacing the nonlogical axiom $\overline{\Delta}$ with $\mathcal{S}'_{\varphi'}$ is $\mathcal{S}_\varphi$.* ∎

3.4. The Empire of a Formula.

The notion of *empire* is defined as the closure of an inductive definition applied to arbitrary proof-structures in **MALL**.

Let $\mathcal{S}$ be a proof-structure in the language $\mathcal{L}_L$ without exponentials.

Definition. The *empire* of a formula-occurrence A in a proof-structure $\mathcal{S}$ is inductively defined as follows (cfr. Girard [1987b], Definition 2.9.3 and Facts 2.9.4, pag 37). We write $e(A)$ for the empire of A.

(i) $A \in e(A)$;

(ii) for any non-axiom link in $\mathcal{S}$, if the conclusion of the link is in $e(A)$, then the premise[s] of the link are in $e(A)$;

(iii) for any axiom $(X_1, \dots, X_n)$ in $\mathcal{S}$ if $X_i \in e(A)$ for some $i \leq n$, then $X_j \in e(A)$ for all $j \leq n$;

(iv) for every one-premise link — i.e., a $\oplus$-, $\bigwedge$- and $\bigvee$-link — and for every $\otimes$-link in $\mathcal{S}$ if one premise of the link is different from A and belongs to $e(A)$, then the conclusion of the link is in $e(A)$;

(v) for any $\sqcup$-, $\sqcap$- and contraction link in $\mathcal{S}$, if all the premises of the link are different from A and all belong to $e(A)$, then the conclusion belongs to $e(A)$.

Definition. (Girard [1987c], II.1, remark 1). Let A and B be formula-occurrences in a proof-structure $\mathcal{S}$. A is the *main door* of $e(A)$. B is a *side door* of $e(A)$ if a link $\frac{B}{C}$ occurs in $\mathcal{S}$, $B \in e(A)$, $B \neq A$ and $C \notin e(A)$. B is a *fake door* of $e(A)$ if $B \in e(A)$ and B belongs to the conclusion of $\mathcal{S}$. A formula-occurrence $\underline{X \otimes X^\perp}$ is called a *ghost door* of $e(A)$ if $\underline{X \otimes X^\perp} \in e(A)$.

Remarks. (i) Clearly a side door must be a premise of a $\sqcup$-link or of a $\sqcap$-link or of a contraction link.

(ii) As pointed out in section (1.1.5), the doors of the empire of a formula A correspond in Natural Deduction to the (open and closed) assumptions relevant to A. For this reason we would prefer the terminology *closed door* and *open door*, for *side door* and *fake door*, but we continue to use Girard's terminology.

Definition. (Girard [1987c] II.1, Definition 2)

$A \emptyset B$ means: $A \notin e(B)$ and $B \notin e(A)$.

$A \sqsubset B$ means: $A \in e(B)$ and $B \notin e(A)$.

$A \Diamond B$ means: $A \in e(B)$ and $B \in e(A)$.

3.5. Proof-Nets.

Conditions are provided that characterize proof-nets, the proof-structures that represent correct reasoning in the fragments of Linear Logic under consideration.

Let $\mathcal{S}$ be a proof-structure for first order **MALL** with conclusion Γ and flagging function φ.

Let $\bigwedge x.A(x)$ and $\bigwedge y.B(y)$ be the conclusions of two $\bigwedge$-links in $\mathcal{S}$ with eigenvariables a and b, respectively. We define the ordering $<^0$ by

Definition. (Girard [1987c] II.3, definition 4) $a <^0 b$ if and only if *the eigenvariable b occurs outside and inside* $e(A(a))$.

Let $<_t$ be the transitive closure of $<^0$.

The *vicious circle* condition (i.e., the condition preventing any vicious circle) is the property

If A and B are the premises of a $\otimes$-link then $A \emptyset B$.

The *connectedness* condition is the property

If A and B are either (1) *among the conclusions or the ghost formula-occurrences of $\mathcal{S}$, or* (2) *the premises of a $\sqcup$-link, then $A \Diamond B$.*

The *box* condition is the property

If A and B are the premises of a $\sqcap$-link, then the main door and the side doors of $e(A)$ [$e(B)$] *are exactly the formula occurrences flagged with A [B].*

The *parameters* condition (cfr Girard [1987c], II.3 Theorem 2) is the property

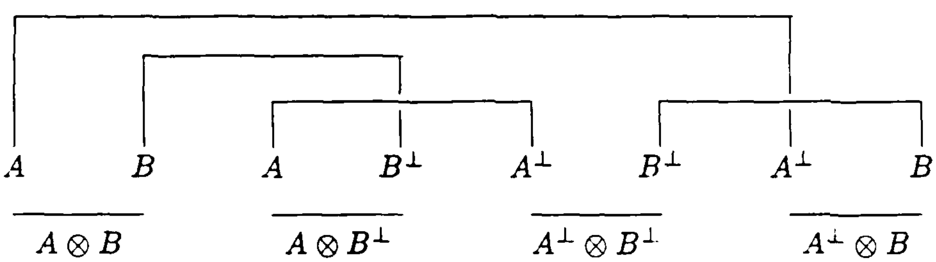
The ordering $<_t$ is strict.

Definitions. (*i*) A proof-structure $\mathcal{S}$ with conclusion Γ in the propositional [first order] fragments $\mathbf{MLL}^-$ or **MLL** is a *proof-net* if it satisfies the vicious circle and connectedness [and parameter] conditions.

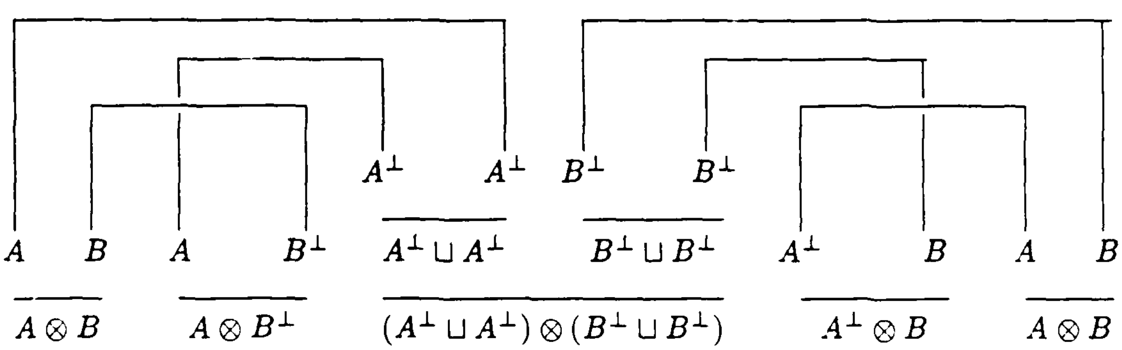
(*ii*) A proof-structure $\mathcal{S}$ with conclusion Γ and flagging function φ in the propositional [first order] fragments **MALL** is a proof net if it satisfies the vicious circle, connectedness, box [and parameter] conditions.

3.5.1. Some Examples.

(1) The following proof-structure violates the vicious circle condition:



(2) The following proof-structure is a proof-net:



(3) The following proof-structure violates the connectedness condition

$$\overline{P, P^\perp}, \qquad \overline{Q, Q^\perp}.$$

(4) The proof-structure with the conclusion

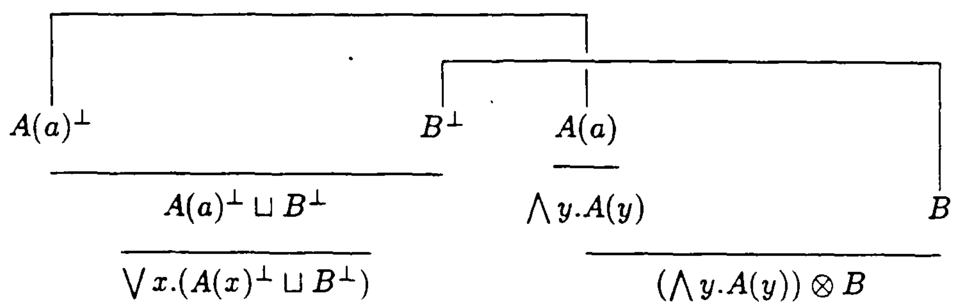
$$\vdash P_1 \otimes Q_1, P_2 \otimes Q_2, (Q_2^\perp \sqcup P_2^\perp) \otimes (Q_1^\perp \sqcup P_1^\perp \sqcup R^\perp),\ R \otimes S,$$
$$(S^\perp \sqcup Q_3^\perp \sqcup P_3^\perp) \otimes (Q_4^\perp \sqcup P_4^\perp),\ P_3 \otimes Q_3,\ P_4 \otimes Q_4$$

and with the axiom-links

$$\overline{P_1, P_1^\perp},\ \overline{Q_1, Q_1^\perp},\ \overline{P_2, P_2^\perp},\ \overline{Q_2, Q_2^\perp},\ \overline{R^\perp, R},\ \overline{S, S^\perp},\ \overline{P_3^\perp, P_3},\ \overline{Q_3^\perp, Q_3},\ \overline{P_4^\perp, P_4},\ \overline{Q_4^\perp, Q_4}$$

also violates the connectedness condition.

(5) The following proof-structure violates the parameters condition:



since $A(a)^\perp \sqcup B^\perp \notin e(A(a))$, hence $a <^0 a$.

(6) Let B and E be 0-ary predicates and let A, C, D be unary predicates, The proof-structure with conclusions

$$\bigvee x.(A(x)^\perp \sqcup B^\perp),\quad B \otimes \bigwedge v.\bigvee x.(C(v) \otimes D(x)),$$
$$(\bigwedge y.D(y)^\perp \otimes A(y)) \otimes E,\quad \bigvee x.(E^\perp \sqcup C(x)^\perp),$$

and with the axiom-links

$$\overline{A(y)^\perp, A(y)},\quad \overline{B^\perp, B},\quad \overline{C(v), C(v)^\perp},\quad \overline{D(y), D(y)^\perp},\quad \overline{E, E^\perp}.$$

violates the parameters condition. Here $v \not<^0 v$ and $y \not<^0 y$, but $v <^0 y$ and $y <^0 v$.

(7) Let A,...,E be as in (6). The proof-structure with conclusions

$$\bigvee x.(A(x)^\perp \sqcup B^\perp),\quad B \otimes \bigwedge v.\bigvee x.(C(v) \otimes D(x)),$$
$$(\bigwedge y.D(y)^\perp \otimes A(y)) \otimes E,\quad E^\perp \sqcup (\bigvee x.C(x)^\perp)$$

and with the same axiom-links as (6) is a first order proof-net. The parameters condition is satisfied, since $v <^0 y$ and $y \not<^0 v$.

3.5.2. Boxes and other Alternative Conditions.

In Girard [1987b], Definition 2.14. and Section 2.4., the notion of *proof-structure with proof-boxes* is introduced. Boxes are described as "synchronization marks in the proof-net". In our terminology, a proof-structure with proof-boxes can be defined as follows.

Definition. A *proof-structure with proof-boxes* is a set $\mathbf{S} = \{\mathcal{S}_1, \ldots, \mathcal{S}_n\}$ of proof-structures $\mathcal{S}_i$ with non-logical axioms and with no flagging function, together with a set of relations $\mathbf{R}$, where each $\mathcal{R} \in \mathbf{R}$ is a relation of the form $\mathcal{R}(\Gamma;\Gamma_1)$ or $\mathcal{R}(\Gamma;\Gamma_1;\Gamma_2)$, and $\Gamma, \Gamma_1, \Gamma_2$ are multisets of formula-occurrences. We write sometimes $\mathcal{R}(\overline{\Gamma};\Gamma_1;\Gamma_2)$, since $\mathcal{R}$ relates the non-logical axiom Γ of some proof-structure with the multiset(s) of conclusions of some other proof-structure(s). Let $\blacktriangleleft$ be the binary relation on $\mathbf{S}$ defined by

$\mathcal{S} \blacktriangleleft^0 \mathcal{S}'$ if and only if for some $\mathcal{R} \in \mathbf{R}$ and some $\Gamma, \Gamma_1, \Gamma_2$, we have either $\mathcal{R}(\Gamma;\Gamma_1)$ or $\mathcal{R}(\Gamma;\Gamma_1;\Gamma_2)$, and moreover $\overline{\Gamma}$ is a non-logical axiom of $\mathcal{S}$, and for $i = 1$ or 2, Γ_i is the multiset of conclusions of $\mathcal{S}'$.

$\mathbf{S}$ and $\mathbf{R}$ must satisfy the conditions that

(i) if $\mathcal{S}_i, \mathcal{S}_j \in \mathbf{S}$ with $i \neq j$, then $\mathcal{S}_i$ and $\mathcal{S}_j$ share no formula-occurrence;

(ii) if $\mathcal{R}(\Gamma;\Gamma_1)$ or $\mathcal{R}(\Gamma;\Gamma_1;\Gamma_2)$, then $\overline{\Gamma}$ is a non-logical axiom of a proof-structure $\mathcal{S} \in \mathbf{S}$ and for each $i \leq 2$, Γ_i is the multiset of the conclusions of some proof-structure $\mathcal{S}_i \in \mathbf{S}$ — i.e., $\mathcal{S} \blacktriangleleft^0 \mathcal{S}_i$;

(iii) for each proof-structure $\mathcal{S} \in \mathbf{S}$ and each non-logical axiom $\overline{\Gamma} \in \mathcal{S}$, there are Γ_1, Γ_2 and $\mathcal{R} \in \mathbf{R}$ such that $\mathcal{R}(\Gamma;\Gamma_1)$ or $\mathcal{R}(\Gamma;\Gamma_1;\Gamma_2)$;

(iv) the transitive closure $\blacktriangleleft$ of $\blacktriangleleft^0$ is a strict partial ordering on $\mathbf{S}$.

The following boxes are introduced in Girard [1987b]: $\perp$-boxes, $\sqcap$-boxes, $\bigwedge$-boxes, Weakening-boxes and !-boxes. Each one of such boxes can be defined as a relation $\mathcal{R} \in \mathbf{R}$, where the multisets in the relation have a certain form, as specified in the following list.

(1) $\mathcal{R}_\perp(\overline{\Gamma, \perp}; \Gamma)$.

(2) $\mathcal{R}_\sqcap(\overline{\Gamma, A \sqcap B}; \Gamma, A; \Gamma, B)$.

(3) $\mathcal{R}_\forall(\overline{\Gamma, \bigwedge x.A}; \Gamma, A(a))$, where a does not occur in Γ.

(4) $\mathcal{R}_{?W}(\overline{\Gamma, ?A}; \Gamma)$.

(5) $\mathcal{R}_{!}(\overline{\Gamma, !A}; \Gamma, A)$, if all the formula-occurrences in Γ are either $\bot$ or have the form $?B$.

Definition. A *proof-structure with proof-boxes* $(\mathbf{S}, \mathbf{R})$ is a *proof-net* (with proof-boxes), if every proof-structure in $\mathbf{S}$ in a proof-net (possibly with non-logical axioms).

The task of eliminating boxes is described in Girard [1987b], section 6. Girard [1987c] accomplishes the elimination of the $\bigwedge$-box. In our treatment, boxes for $\bot$, $\sqcap$ and $\bigwedge$ are replaced by relationships beween empires. The detail will become apparent from the proof of the Sequentialization Theorem for Propositional **MLL**, **MALL** and for First Order **MALL**.

3.6. Structure Theorem for Propositional MALL.

A proof is given that inductively defined proof-structures are proof-nets.

Facts. *Let $\mathcal{S}$ be a proof-structure in first order* **MALL** *of the form*

$$\begin{array}{cccc} \mathcal{S}_1 & & & \mathcal{S}_2 \\ \Pi & A & B & \Lambda \\ & \multicolumn{2}{c}{\overline{\quad A \otimes B \quad}} & \end{array}$$

where $\mathcal{S}_1$ and $\mathcal{S}_2$ share no formula-occurrences.

(i) *If $X \in \mathcal{S}_1[or\ X \in \mathcal{S}_2]$ and $A \otimes B \notin e(X)$, then $\mathcal{S}_2 \cap e(X) = \emptyset$ [or $\mathcal{S}_1 \cap e(X) = \emptyset$].*

(ii) *For $i = 1, 2$ and for all $X, Y \in \mathcal{S}_i$, $Y \in e(X)$ in $\mathcal{S}_i$ if and only if $Y \in e(X)$ in $\mathcal{S}$.*

Proof. Notice that the indicated link is the only connection between $\mathcal{S}_1$ and $\mathcal{S}_2$. (i) follows immediately from this remark. (ii) Suppose $X, Y \in \mathcal{S}_1$ and consider any computation of $Y \in e(X)$. If such computation reaches a formula in $\mathcal{S}_2$ then it must exit and re-enter $\mathcal{S}_1$ through A. Hence the part of the computation involving formulas outside $\mathcal{S}_1$ can be omitted. ∎

Lemma. *Let $\mathcal{S}_\varphi$ be an inductive proof-structure with conclusion Γ in first order* **MALL**. *If A, B are in $\Gamma \cup$* **CUT**, *then $e(A) = \mathcal{S}_\varphi = e(B)$.*

Proof. By induction on the definition of $\mathcal{S}$. If A and B belong to an axiom link, then the result is immediate.

If $\mathcal{S}$ results from $\mathcal{S}_1$ and $\mathcal{S}_2$ by an application of clause (2), then $\mathcal{S}$ has the form

$$\begin{array}{c} \mathcal{S}_1 \qquad\qquad\qquad \mathcal{S}_2 \\ C_1, \dots, C_p, \dfrac{C \qquad D}{C \otimes D} D_1, \dots, D_q \end{array}$$

and by induction hypothesis we have $e(C_i) = \mathcal{S}_1 = e(C)$ and $e(D_j) = \mathcal{S}_2 = e(D)$. Clearly in $\mathcal{S}$ we still have $e(C) \subset e(C_i)$ and $e(D) \subset e(D_j)$. In addition, by induction on $e(C)$ we obtain $e(C) \subset e(C \otimes D)$ and $e(C) \subset e(D_j)$; also by induction on $e(D)$ we obtain $e(D) \subset e(C \otimes D)$ and $e(D) \subset e(C_i)$. In conclusion,

$$\mathcal{S} = e(C) \cup \{C \otimes D\} \cup e(D) = e(C_i) = e(C \otimes D) = e(D_j).$$

If $\mathcal{S}$ results from $\mathcal{S}_1$ by an application of clause $(3) - (6)$ then the result is immediate from the induction hypothesis.

If $\mathcal{S}$ results from $\mathcal{S}_1$ and $\mathcal{S}_2$ by an application of clause (7), then $\mathcal{S}$ is $\mathcal{S}_1 \cup \mathcal{S}_2$ with the additional links

$$\frac{C_1' \quad C_1''}{C_1} \dots \frac{C_p' \quad C_p''}{C_p} \quad \frac{C \quad D}{C \sqcap D}$$

and by induction hypothesis we have $e(C_i') = \mathcal{S}_1 = e(C)$ and $e(C_j'') = \mathcal{S}_2 = e(D)$. Clearly in $\mathcal{S}$ we still have $\mathcal{S}_1 \subset e(C)$, $\mathcal{S}_1 \subset e(C_i')$, $\mathcal{S}_2 \subset e(D)$ and $\mathcal{S}_2 \subset e(C_j'')$, for $i, j \leq p$. In addition in $\mathcal{S}$ by induction on $e(C_i')$ we obtain $e(C_i') \subset e(C_i)$; also by induction on $e(C_i'')$ we obtain $e(C_i'') \subset e(C_i)$, thus $C_1, \dots, C_p, C \sqcap D \in e(C_i)$, by clause (v) in the definition of empire. Hence

$$\mathcal{S} = \mathcal{S}_1 \cup \{C_1, \dots, C_p, C \sqcap D\} \cup \mathcal{S}_2 = e(C_i).$$

Similarly we show that $\mathcal{S} = e(C \sqcap D)$. ∎

Structure Theorem. *If* $\mathcal{S}_\varphi$ *is an inductive proof-structure for* **MLL**$^-$, **MLL** or **MALL** *then it a proof-net.*

Proof. We argue by induction on the definition of $\mathcal{S}$ and we use the notation of the previous Lemma.

(i) (*Vicious Circle condition*). We inductively prove $e(A) \cap e(B) = \emptyset$, which implies the condition. Let A be B are premises of a $\otimes$-link. Suppose $\mathcal{S}$ is obtained from $\mathcal{S}_1$

and $\mathcal{S}_2$ by an application of clause (2). The proof-structures $\mathcal{S}_1$ and $\mathcal{S}_2$ associated with the immediate subderivations satisfy the vicious circle condition and do not share any formula-occurrence and the link

$$\frac{C \quad D}{C \otimes D}$$

is the only connection between $\mathcal{S}_1$ and $\mathcal{S}_2$ in $\mathcal{S}$.

Let $A \in \mathcal{S}_1$ — if $A \in \mathcal{S}_2$, the argument is similar. Let X be a formula occurrence X in $\mathcal{S}_2$. If $X \in e(A)$, then by the Fact (i) above $C \otimes D \in e(A)$. Also the first step in the computation of $e(A)$ reaching a formula occurrence outside $\mathcal{S}_1$ must be an application of clause (iv) in the definition of empire, to conclude that $C \otimes D = A \otimes B \in e(A)$ from the fact that $C \in e(A)$.

If on the one hand A is C, then this step is not admissible, so $e(A) \cap \mathcal{S}_{\mathcal{D}_2} = \emptyset$. Moreover in this case B is D and for the same reason $e(B) \cap \mathcal{S}_{\mathcal{D}_1} = \emptyset$, thus $e(A) \cap e(B) = \emptyset$.

If on the other hand A is different from C, then $A \otimes B$ is also in $\mathcal{S}_1$ and by induction hypothesis $e(A) \cap e(B) = \emptyset$ in $\mathcal{S}_1$, in particular, $C \in e(A)$ if and only if $C \notin e(B)$. By Fact (ii) above, the same holds in $\mathcal{S}$. Therefore the step from C to $C \otimes D$ cannot be available simultaneously in a computation of $e(A)$ and of $e(B)$.

If $\mathcal{S}$ is obtained by an application of clauses (3)–(6) then the result is immediate from the induction hypothesis. Finally, if $\mathcal{S}$ is obtained by an application of clause (7), then the result follows from the induction hypothesis and the box condition proved below.

(ii) (*Connectedness condition*). We inductively prove $e(A) = e(B)$, which implies the condition. The above Lemma yields the result if A and B are elements of Γ. Let A and B be the premises of a $\sqcup$-link.

Suppose $\mathcal{S}$ is obtained from $\mathcal{S}_1$ and $\mathcal{S}_2$ by an application of clause (2) and that $A \sqcup B$ occurs in $\mathcal{S}_1$ — if $A \sqcup B$ occurs in $\mathcal{S}_2$, then the argument is similar.

By induction hypothesis $e(A) = e(B)$ in $\mathcal{S}_1$, in particular, $C \in e(A)$ if and only if $C \in e(B)$. By Fact (ii) above, the same holds in $\mathcal{S}$.

It follows that in $\mathcal{S}$, $C \otimes D \in e(A)$ if and only if $C \otimes D \in e(B)$ and by induction on $e(D)$, for all X in $\mathcal{S}_2$, $X \in e(A)$ if and only if $X \in e(B)$. We conclude that $e(A) = e(B)$ in $\mathcal{S}$ too.

If $\mathcal{S}$ is obtained by an application of clauses (3)–(6) then the result is immediate from the induction hypothesis. Finally, if the last inference is a $\sqcap$-rule, then the result follows from the induction hypothesis and the box condition.

(iii) (*Box condition*). The cases when $\mathcal{S}$ results from an application of clauses (1) – (6) is straightforward.

Suppose $\mathcal{S}(\Gamma)$ results from $\mathcal{S}_1(\Gamma')$ and $\mathcal{S}_2(\Gamma'')$ by clause (7) where

$$\Gamma' = C_1', \ldots, C_m', C, \ , \Gamma'' = C_1'', \ldots, C_m'', D, \ \Gamma = C_1, \ldots, C_m, C \sqcap D$$

and $C_i' \sim C_i \sim C_i''$. Moreover, $\mathcal{S}_1(\Gamma')$ and $\mathcal{S}_2(\Gamma'')$ do not share any formula-occurrence, hence the $\sqcap$- and Contraction links

$$\frac{C \quad D}{C \sqcap D} \qquad \frac{C_1' \quad C_1''}{C_1}, \quad \ldots \quad , \frac{C_m' \quad C_m''}{C_m}$$

are the only links in $\mathcal{S}$ connecting $\mathcal{S}_1$ and $\mathcal{S}_2$.

By induction hypothesis $\mathcal{S}_1(\Gamma')$ and $\mathcal{S}_2(\Gamma'')$, satisfy the box condition. We need to show that in $\mathcal{S}$ the formulas in Γ' are all the doors of $e(C)$ and the formulas in Γ'' are all the doors of $e(D)$. This follows from the connectedness condition for $\mathcal{S}_1$ and $\mathcal{S}_2$ and from the following fact.

Fact: *In $\mathcal{S}$ for all $Y \in \Gamma$, $Y \notin e(C)$ and $Y \notin e(D)$.*

Let $Y \in \Gamma$, suppose $Y \in e(C)$ and consider a computation of $Y \in e(C)$. Since $Y \notin \mathcal{S}_1$, the computation must exit $\mathcal{S}_1$ first through a link $\frac{V \quad W}{Z}$, where V_1 belongs to $\mathcal{S}_1$. and W belongs to $\mathcal{S}_2$. Since the link in question is either a $\sqcap$- or a contraction link, we must infer $Z \in e(C)$ from $V \in e(C)$ and $W \in e(C)$ by an application of clause (v) in the definition of empire. But $W \notin \mathcal{S}_1$ and this contradicts the assumption that Z is the first formula-occurrences outside $\mathcal{S}_1$ to be reached in the computation.

(iv) (*Parameters condition*) We need to show that every inductive proof-structure can be transformed into one satisfying the following property (cfr. the *pure parameter property* in sequent calculi):

Pure Parameter Property. *Suppose*

$$\frac{A(a)}{\bigwedge x.A(x)}$$

is a link of $\mathcal{S}_\varphi(\Gamma)$. Then a does not occur in the conclusions of $\mathcal{S}$.

Assuming the Pure Parameter property, we prove the parameters condition. All the propositional clauses (1) – (3), (6) – (7) and also clause (5) are easy. Suppose $\mathcal{S}_\varphi$ is obtained from $\mathcal{S}'_{\varphi'}$ by an application of those clauses. By the Pure Parameter Property we may regard all eigenvariables occurring in the conclusion of $\mathcal{S}'_{\varphi'}$ as special constants; thus the $<_t$ relation is the same in $\mathcal{S}_\varphi$ and in $\mathcal{S}'_{\varphi'}$.

Suppose that $\mathcal{S}_\varphi$ results from $\mathcal{S}'_{\varphi'}$ by an application of clause (4) introducing a link

$$\frac{B(b)}{\bigwedge x.B(x).}$$

Let $<'_t$ be the transitive closure of the relation $<^0$ defined over $\mathcal{S}'_{\varphi'}$. By induction hypothesis, $<'_t$ is strict. We need to show that if $<_t$ is the transitive closure of the relation $<^0$ defined over $\mathcal{S}_\varphi$, then $<_t$ is strict. Since a $\bigwedge$-link is unary, all the empire relationships between formulas different from $B(b)$ and $\bigwedge x.B(x)$ are the same in $\mathcal{S}'_{\varphi'}$ and in $\mathcal{S}_\varphi$. Therefore to show that $<^t$ is strict, it is enough to show that we cannot have $a <_t b <^0 a$, for some a. By part (ii), $\mathcal{S}'_{\varphi'} \subset e(B(b))$. By the pure parameter property, if

$$\frac{A(a)}{\bigwedge x.A(x)}$$

is a link in $\mathcal{S}'_{\varphi'}$, then a does not occur in $B(b)$ and in $\bigwedge x.B(x)$. Therefore we cannot have $b <^0 a$ in $\mathcal{S}_\varphi$, and $<_t$ is strict. To finish the proof of the theorem we need to guarantee the Pure Parameters Property for any inductive proof-structure.

If $\mathcal{S}$ results from an application of clause (4), introducing a new link

$$\frac{A(a)}{\bigwedge x.A(x),}$$

then the restriction on a yields the desired property. The property is preserved by clauses (3) and (5). At any application of clauses (2) and (7) we must guarantee that every eigenvariable b occurring in the conclusions of $\mathcal{S}'$ is different from all eigenvariables a occurring in the premise of any $\bigwedge$-link of $\mathcal{S}''$ (and viceversa). If $a = b$, we may rename the eigenvariable b. The renaming is possible, since, by induction hypothesis a does not occur in the conclusions of $\mathcal{S}''$. A similar argument applies if $\mathcal{S}$ results by an application of clause (6). ■

3.7. A map from Derivations to Proof-Nets.

A map is given from derivations to inductively defined proof-structures for the fragments of **LL** under consideration.

The proof of the following lemma is straightforward from the remark that we can always permute an application of the $\perp$-rule above any other inference (including the !-rule).

Lemma. *Every derivation $\mathcal{D}'$ in* **LL** *can be transformed into a derivation $\mathcal{D}$ where above every application of the $\perp$-rule there is either an axiom or another application of the $\perp$-rule.* ■

Definition. Let $\mathcal{D}$ be a derivation of $\vdash \Gamma$ in propositional or first order (i) **MLL$^-$**, (ii) **MLL**, (iii) **MALL** By induction on the length of $\mathcal{D}$ we construct an *inductive proof-structure $\mathcal{S}_\mathcal{D}$ associated with $\mathcal{D}$* with conclusion Γ and a many-to-one map $\pi : \mathcal{D} \rightarrow \mathcal{S}_\mathcal{D}$ sending formula-occurrences of $\mathcal{D}$ to formula-occurrences of $\mathcal{S}$, active and principal formulas of a $\circ$- rule of $\mathcal{D}$ to premises and conclusion of a $\circ$-link in $\mathcal{S}_\mathcal{D}$.

Propositional fragments, case (i): in correspondence to a logical axioms, to the $\otimes$-rule and to the $\sqcup$-rule, we use clauses (1), (2) and (3) respectively. The same proof-structure corresponds to the premise and to the conclusion of an application of the Exchange rule.

case (ii): first we apply the above Lemma and then we use clause $(1)'$, in correspondence to a **1**- or logical axiom followed by a sequence of application of the $\perp$-rule. The other rules are handled as in case (i).

case (iii): in correspondence to the $\top$-axiom we use clause $(1)''$; in correspondence to the $\oplus$-rule and the $\sqcap$-rule we use clauses (6) and (7), respectively. The other rules are handled as in case (ii).

First order fragments, cases $(i) - (iv)$: in correspondence to the $\bigwedge$- and $\bigvee$-rule we use clauses (4) and (5), respectively. The other axioms and rules are handled as in the propositional cases.

4. Main Theorem.

Structure Theorem. *If $\mathcal{D}$ is a derivation of $\vdash \Gamma$ in* MLL$^-$, MLL *or* MALL *then we can associate with $\mathcal{D}$ a proof-net $\mathcal{S}_{\mathcal{D}}$ with conclusion Γ.*

Sequentialization Theorem. *If $\mathcal{S}$ is a proof-net with conclusion Γ for* MLL$^-$, MLL, *or* MALL *then we can find a derivation $\mathcal{D}$ of $\vdash \Gamma$ in the corresponding fragment such that the associated proof-net $\mathcal{S}_{\mathcal{D}}$ is $\mathcal{S}$.*

The above form of the Structure Theorem follows from the more general Theorem in section (3.6) by the map π defined in section (3.7). The sections (4.1 - 4.6) are devoted to the proof of the Sequentialization Theorem.

4.1. Computations of the Empire of a Formula.

The inductive definition of empire allows us to make arguments *by induction on the definition of empire* of some formula. In addition, we will need the fact that the empire of some formulas can be computed only in a certain way. Therefore, it is desirable to provide a formal description of the computation of (or search through) the empire of a formula in a structure. The inductive trees considered below are easy instances of Inductively Presented Systems, extensively studied in Feferman [1982].

An algorithm computing $e(A)$ in $\mathcal{S}$ consists of a succession of steps through the links of $\mathcal{S}$ corresponding to applications of clauses $(i) - (v)$ in the definition of empire.

(i) Start at A;

(ii) $\uparrow \dfrac{X_i}{X_1 \circ X_2} \qquad \uparrow \dfrac{X}{Qy.X}$

(iii) $\overline{\cdots X_j \underset{\longrightarrow}{\cdots} X_k \cdots}$

(iv) $\downarrow \dfrac{X_i}{X_1 \otimes X_2} \qquad \downarrow \dfrac{X_i}{X_1 \oplus X_2} \qquad \downarrow \dfrac{X}{Qy.X} \qquad X_i, X \neq A, i = 1, 2;$

(v) $\Downarrow \dfrac{X_i}{X_1 \sqcap X_2} \qquad \Downarrow \dfrac{X_i}{X} \qquad \Downarrow \dfrac{X_i}{X_1 \sqcup X_2} \qquad X_i \neq A, i = 1, 2.$

where $\circ$ is either $\otimes$ or $\sqcup$ or $\sqcap$ or $\oplus$ and Q is either $\bigwedge$ or $\bigvee$. A step $\Downarrow$ differs from the others since its performance depends on a condition, namely that both X_1 and X_2 have been already reached in the computation. Thus the computation of the empire of A consists of sequences of steps $(i) - (v)$ that are available given the form of $\mathcal{S}$ and that are admissible according to the definition of empire. (Sometimes we write $(\uparrow^i_\circ)$, $(\downarrow^i_\circ)$ and $(\Downarrow^i_\circ)$, if the step is across a $\circ$-link.) We define the *tree of admissible empire-sequences* $\tau^B_{[A]}$ from A to B as follows. Every node in $\tau^B_{[A]}$ is labeled with a formula-occurrence of $\mathcal{S}$ and every edge is identified with one of the steps $(i) - (v)$ above. The root of $\tau^B_{[A]}$ is labeled by B and the leafs are labeled by A (we think of $\tau^B_{[A]}$ as 'growing downwards'). '$*$' is used here for concatenation of sequences.

Definition. (I) The sequence $\langle A \rangle$ is a tree $\tau^A_{[A]}$;

(II) if $\tau^Y_{[A]}$ is a tree of admissible empire-sequences, $\frac{\cdots X_i \cdots}{Y}$ is any link in $\mathcal{S}$, then the result of extending every empire-sequence α of $\tau^Y_{[A]}$ to $\alpha * \langle X_i \rangle$ is a tree of admissible empire-sequences $\tau^X_{[A]}$ (*step* $(\uparrow^i)$).

(III) if $\tau^Y_{[A]}$ is a tree of admissible empire-sequences and Y and X are the j-th and the k-th *elements of an axiom, respectively, then the result of extending every* empire-sequence α of $\tau^Y_{[A]}$ to $\alpha * \langle X \rangle$ is a tree of admissible empire-sequences $\tau^X_{[A]}$ (*step* $(\overrightarrow{_{j,k}})$);

(IV) if $\tau^Y_{[A]}$ is a tree of admissible empire-sequences, $\frac{\cdots Y \cdots}{X}$ is a $\otimes$-link or a one-premise link, where $Y \neq A$ and Y is the i-th premise, then the result of extending every empire-sequence α of $\tau^Y_{[A]}$ to $\alpha * \langle X \rangle$ is a tree of admissible empire-sequences $\tau^X_{[A]}$ (*step* $(\downarrow^i)$);

(V) let $\tau^{Y_1}_{[A]}$ and $\tau^{Y_2}_{[A]}$ be a trees of admissible empire-sequences with $Y_1 \neq A \neq Y_2$; if $\frac{Y_1 \; Y_2}{X}$ is a $\sqcap$-link or a $\sqcup$-link or contraction link, let ${\tau^i}^X_{[A]}$ be the result of extending every empire-sequence α of $\tau^{Y_i}_{[A]}$ to $\alpha * \langle X \rangle$. Then the union of ${\tau^1}^X_{[A]}$ and ${\tau^2}^X_{[A]}$ is a tree of admissible empire-sequences $\tau^X_{[A]}$ (*step* $(\Downarrow^i)$).

Remark. $B \in e(A)$ *if and only if there is a tree of admissible empire-sequences* $\tau^B_{[A]}$ *from* A *to* B.

The nodes of $\tau^B_{[A]}$ are labeled with formula-occurrences in $\mathcal{S}$ and the same formula-occurrence may appear several times in it as a part of different subcomputations.

We want a normal form for computations. The following transformations constitute *reductions* on a tree of admissible empire-sequences. Let $\tau^B_{[A]}$ be such a tree. (1) A pairs of successive steps $(\underset{i,j}{\rightarrow})$ $(\underset{j,k}{\rightarrow})$ in $\tau^B_{[A]}$ can be replaced by a single step $(\underset{i,k}{\rightarrow})$. (2) If one formula-occurrence X in $\mathcal{S}$ labels two nodes within the same empire-sequence, let $\tau'^X_{[A]}$ and $\tau''^X_{[A]}$ be the subtrees of $\tau^B_{[A]}$ which represent the subcomputations ending with the first and the second passage through X. Then the result of replacing $\tau''^X_{[A]}$ with $\tau'^X_{[A]}$ in $\tau^B_{[A]}$ is still a tree $\tau'^B_{[A]}$ of admissible empire sequences.

Definition. (i) An empire-sequence is *pure* if all its nodes are labeled with different formula-occurrences in $\mathcal{S}$.

(ii) A tree τ is *normal* if every empire-sequence in it is pure.

Each one of the above reductions reduces the size of the tree. If $\tau^B_{[A]}$ is finite, then after a finite number of reductions we obtain a normal $\tau'^B_{[A]}$.

4.1.1. Elementary Properties of the Empire of a Formula.

The following properties follow immediately from the definition of empire.

Quasi-Transitivity Lemma. *Let $B \in e(A)$ and $C \in e(B)$. If in some computation of $C \in e(B)$ there is no step of the form $(\downarrow)$ or $(\Downarrow)$ through $\frac{A}{Z}$ for some Z, then $C \in e(A)$.*

Proof. Let $\tau^B_{[A]}$ represent a computation that $B \in e(A)$. If C occurs in $\tau^B_{[A]}$ then a computation of $C \in e(A)$ is represented by the subtree $\tau'^C_{[A]}$ of $\tau^B_{[A]}$, i.e., the tree whose top node is labeled by the occurrence of C in question.

Suppose C does not occur in $\tau^B_{[A]}$. If $\tau^C_{[B]}$ represents a computation of $C \in e(B)$ with the prescribed property, then the result of replacing each bottom node (leaf) of $\tau^C_{[B]}$ with a copy of $\tau^B_{[A]}$ represents a computation of $C \in e(A)$. ■

Corollary. *If $B \sqsubset A$, then $e(B) \subsetneq e(A)$.* ■

We will use the Corollary without mention.

Proposition. *Let $\tau^B_{[A]}$ be a normal tree of admissible empire-sequences through $\mathcal{S}$.* (i) *If $B \not\prec A$ and the last step in $\tau^B_{[A]}$ is of the kind $(\uparrow)$, then there is a a formula-occurrence $C = C_1 \otimes C_2$ such that $B \prec C$ and every empire-sequence in $\tau^B_{[A]}$ ends with a pair of steps $(\downarrow^j_\otimes)\frac{C_j}{C}, (\uparrow^i_\otimes)\frac{C_i}{C}$ with $j \neq i$, followed by a sequence of steps of*

the form $(\uparrow)$. E.g., in the case $C = C_1 \otimes C_2$ and $B \prec C_2$ the computation proceeds as follows:

$$\begin{array}{ccc} & B & \\ & \vdots & (\uparrow) \\ (\downarrow^1_\otimes) & \dfrac{C_1 \quad C_2}{C_1 \otimes C_2} & (\uparrow^2_\otimes). \end{array}$$

(ii) *If the last step in* $\tau^B_{[A]}$ *is of the kind* $(\downarrow)$ *or* $(\Downarrow)$, *then for every branch of* $\tau^B_{[A]}$ *there exists a formula-occurrence* X *belonging to an axiom link such that* $X \prec B$ *and after the node* X *the branch ends with a sequence of steps of the kind* $(\downarrow)$ *or* $(\Downarrow)$. I.e., the computation has the form

$$\begin{array}{cc} \overline{\cdots \underrightarrow{Y \cdots X} \cdots} & \\ \vdots & (\downarrow, \Downarrow) \\ B & \end{array}$$

Proof. (i) Consider any empire-sequence γ of $\tau^B_{[A]}$. If all steps of γ were of the form $(\uparrow)$, then for all formula X reached by γ we would have $B \prec X$, thus also $B \prec A$. Consider the last step of γ which is not of the form $(\uparrow)$ and let C be the formula reached by this step. C cannot belong to an axiom, because after an axiom we cannot have a step $(\uparrow)$. If C was $C_1 \oplus C_2$, $\bigwedge x.C_1$ or $\bigvee x.C_1$, then γ would contain two successive steps

$$(\downarrow^i) \quad \frac{C_i}{C} \quad (\uparrow^i)$$

reaching twice the same formula-occurrence C_i of $\mathcal{S}$. Suppose C is $C_1 \sqcup C_2$, $C_1 \sqcap C_2$ or a premise of a contraction link. Then in $\tau^B_{[A]}$ there is an empire-sequence γ' containing two successive steps

$$(\Downarrow^i) \quad \frac{C_i}{C} \quad (\uparrow^i)$$

reaching twice the same formula-occurrence C_i of $\mathcal{S}$. Therefore C is $C_1 \otimes C_2$ and the steps before and after the node C have the form $(\downarrow^j_\otimes)$ $(\uparrow^i_\otimes)$ with $i \neq j$, as required.
(ii) For all empire-sequences γ, not all steps of γ are of the form $(\downarrow)$ or $(\Downarrow)$, since form A we can only have a step $(\uparrow)$ or $(\underset{j,k}{\rightarrow})$. For any γ, consider the last step which is not of the kind $(\downarrow)$ or $(\Downarrow)$. By an argument similar to that of part (i) this step can only be of kind $(\underset{j,k}{\rightarrow})$. If X is the element of the axiom link reached at the end of this step, then $X \prec B$, as required. ∎

4.2. Consequences of the Vicious Circle Condition.

Properties of the relation $\sqsubset$ are considered in structures that satisfy the basic consistency requirement (*vicious circle condition*).

Proposition. *Let $\mathcal{S}$ be any proof-structure satisfying the vicious circle condition.*
(i) *Let C be $C_1 \otimes C_2$, the conclusion of a link*

$$\frac{C_1 \quad C_2}{C_1 \otimes C_2.}$$

If for some normal computation of $C \in e(A)$ the last step in $\tau^C_{[A]}$ does not have the form $(\downarrow^i_\otimes)$, then $C_i \sqsubset A$.

(ii) *If $B \prec A$, then $B \sqsubset A$.*

(iii) *If for some normal computation of $B \in e(A)$ the last step in $\tau^B_{[A]}$ has the form $(\uparrow)$ then $B \sqsubset A$.*

Proof. (*i*) If $\tau^C_{[A]}$ is a normal tree satisfying the assumption, then no step in $\tau^C_{[A]}$ has the form $(\downarrow)^i_\otimes$ through $\frac{\cdots C_i \cdots}{C}$. Clearly $C_i \in e(A)$, for all i. If $A \in e(C_i)$, then by the Quasi-Transitivity Lemma (Section 4.1.1) $C_j \in e(C_i)$ for $j \neq i$, contradicting the vicious circle condition. Thus $C_i \sqsubset A$.

(*ii*) Assuming $B \prec A$, $B \in e(A)$ is clear. Assume $A \in e(B)$. The following fact can be easily proved by induction on the logical complexity of X:

Fact. *If X is a formula-occurrence such that $B \prec X$ and for no multiplicative conjunction C we have $B \prec C \prec X$, then no normal computation of $X \in e(B)$ can end with a step $(\downarrow)$ or $(\Downarrow)$*

If on one hand no normal computation $\tau^A_{[B]}$ ends with a step $(\uparrow)$, then there is a multiplicative conjunction C, with $B \prec C \prec A$ and since $A \in e(B)$ we have also $C \in e(B)$.

If on the other hand the last step of a normal computation $\tau^A_{[B]}$ ends with a step $(\uparrow)$, Then since $A \not\prec B$, we can apply part (i) of the Proposition in Section (4.1.1) and find a multiplicative conjunction C with $A \prec C$ and $C \in e(B)$.

Thus we may choose C to be the smallest (with respect to $\prec$) multiplicative conjunction such that $B \prec C$ and $C \in e(B)$, say $C = C_1 \otimes C_2$ and $B \prec C_2$. Clearly, $B \in e(C_2)$ and $C_1 \in e(B)$. Fix $\tau^B_{[C_2]}$ and let $\tau^{C_1}_{[B]}$ be normal. By the above fact

and the choice of C, $\tau^{C_1}_{[B]}$ cannot contain a step $\downarrow \frac{C_2}{C_1 \otimes C_2}$. Therefore, by the Quasi-Transitivity Lemma (Section 4.1.1), $C_1 \in e(C_2)$, contradicting the vicious circle condition.

(*iii*) The case $B \prec A$ is case (*ii*), thus suppose $B \not\prec A$. Then by part (*i*) of the Proposition in Section (4.1.1), there exists a multiplicative conjunction C, say $C = C_1 \otimes C_2$, with $C \in e(A)$ and $B \prec C$, say $B \prec C_2$ and there is a normal tree $\tau^{C}_{[A]}$, which does not contain any step $\downarrow \frac{C_2}{C_1 \otimes C_2}$.

Now suppose $A \in e(B)$. By part (*ii*), $e(B) \subset e(C_2)$ thus $A \in e(C_2)$. It follows from the Quasi-Transitivity Lemma that $C \in e(C_2)$, thus $C_1 \in e(C_2)$, contradicting the vicious circle condition. ∎

Define a relation $\lhd$ between the conclusions of the $\sqcap$-links of S by letting

$$A' \sqcap B' \lhd A \sqcap B =_{df} A' \sqcap B' \in e(A) \text{ or } A' \sqcap B' \in e(B).$$

Box Lemma *Let S be a structure for* **MALL** *with flagging function φ satisfying the vicious circle and the box conditions and let $\frac{A_1 \quad A_2}{A_1 \sqcap A_2}$ be a link in S.*

(i) *If $X \in e(A_i)$ then $e(X) \subset e(A_i)$.*

(ii) $e(A_1) \cap e(A_2) = \emptyset$.

(iii) $\lhd$ *is a strict partial ordering.*

Proof. (*i*) Let A be A_1 or A_2, say A_1 and let Δ_A be the set of formula-occurrences flagged with A. By the box conditions, these are all the doors of $e(A)$ that are not a ghost door. Let $Y \in e(X)$ and consider a normal computation of $Y \in e(X)$. If $Y \notin e(A)$, then any empire-sequence in $\tau^{Y}_{[X]}$ must exit $e(A)$ through some door D by a step $\downarrow \frac{D \quad E}{F}$. Also $\tau^{Y}_{[X]}$ must contain a subtree $\tau^{E}_{[X]}$ and since $E \notin e(A)$ any empire-sequence in $\tau^{E}_{[X]}$ must exit $e(A)$ through some door D' by a step $\downarrow \frac{D' \quad E'}{F'}$. Moreover F' cannot be the same formula-occurrence as F, because $\tau^{Y}_{[X]}$ is normal; thus D is distinct from D' too. By repeating this argument we obtain an infinite sequence $D, D', \ldots$ of distinct doors of $e(A)$, a contradiction since Δ_A is finite.

(*ii*)) Let $X \in e(A_1) \cap e(A_2)$. Let D be the smallest formula-occurrence with $X \prec D$ such that D is a door either of $e(A_1)$ or of $e(A_2)$, say of $e(A_1)$. By the box condition, D belongs to a $\sqcap$- or Contraction link

$$\frac{D \quad E}{F},$$

where and E is a door of $e(A_2)$. But $D \in e(A_2)$, thus E cannot be a door of $e(A_2)$, a contradiction. Therefore $e(A_1) \cap e(A_2) = \emptyset$.

(iii) If $A_0 \sqcap B_0 \triangleleft \ldots A_n \sqcap B_n \triangleleft A_0 \sqcap B_0$, then $e(C_0) \subsetneq e(C_1) \subsetneq \ldots e(C_n) \subsetneq e(A_0)$ or $e(C_0) \subsetneq e(C_1) \subsetneq \ldots e(C_n) \subsetneq e(B_0)$ where C_i is either A_i or B_i, by parts (i) and (ii) and by the above Proposition, part (ii). ■

4.3. Tiling Lemma and Substructure Theorem for MLL.

The main technical results relevant to proof-net for **MLL** are presented. Given the four alternatives

$$A \emptyset B \qquad A \sqsubset B \qquad B \sqsubset A \qquad A \Diamond B,$$

the *Tiling Lemma* shows that in a proof net

$$A \emptyset B \qquad \text{if and only if} \qquad e(A) \cap e(B) = \emptyset.$$

The Tiling Lemma is fundamental for the Sequentialization Theorem. An interesting result follows: (*Substructure Theorem*) *Let $\mathcal{S}$ be a proof-net for* **MLL** *and let A occur in $\mathcal{S}$. Then $e(A)$ is a substructure of $\mathcal{S}$ that is also a sub-net.* Given the Substructure Theorem, the informal meaning of Tiling Lemma may be stated as follows: if some X is deductively relevant to both A and B, then we can find a deduction of B in which either A or $\neg A$ occurs as an assumption (and viceversa, with A and B exchanged).

Proposition. *Let $\mathcal{S}$ be a proof-net for* **MLL**. *The following are equivalent:*

(i) $B \sqsubset A$

(ii) *for some door D of $e(B)$ and some normal computation of $D \in e(A)$, the last step in $\tau^{D}_{[A]}$ has the form* $(\uparrow)$

(iii) *for some normal computation of $B \in e(A)$ the last step in $\tau^{B}_{[A]}$ has the form* $(\uparrow)$.

Proof. $(i) \Rightarrow (ii)$: let $\tau^{B}_{[A]}$ be a normal computation of $B \in e(A)$. Since $A \notin e(B)$, each empire-sequence in $\tau^{B}_{[A]}$ reaches for the first time some $D \in e(B)$. Fix any empire-sequence and consider the normal subtree $\tau^{D}_{[A]}$.

$(iii) \Rightarrow (i)$ is part (iii) of the Proposition in Section (4.2).

$(ii) \Rightarrow (iii)$. If D is the main door of $e(B)$, then we are finished. Otherwise, the last step in $\tau^{D}_{[A]}$ crosses a $\sqcup$-link $\uparrow \frac{D}{E}$, say E is $C \sqcup D$. By part (iii) of the Proposition in Section (4.2), $e(D) \subsetneq e(A)$. By the Connectedness condition we have $e(C) = e(D)$.

Consider any normal computation of $D \in e(C)$ and let $\tau^{D}_{[C]}$ be the corresponding tree of admissible empire-sequences. Since $e(C) \cap e(B) \neq \emptyset$ and $C \notin e(B)$, any empire-sequence in $\tau^{D}_{[C]}$ must enter $e(B)$ through some door D', either the main door — and in this case we are done — or a side door D' crossing a $\sqcup$-link $\uparrow \frac{D'}{E'}$, say with $E' = C' \sqcup D'$. Again we have $e(D') \subsetneq e(C)$ (Proposition (iii), 4.2) and $e(C') = e(D')$ (by connectedness). Let C_0, C_1, C_2 be A, C C' and let D_0, D_1, D_2 be B, D, D', respectively.

Proceeding in this way, we obtain a sequence D_0, D_1,... of doors of $e(B)$ and a sequence C_0, C_1,... such that

$$\ldots \subsetneq e(C_1) \subsetneq e(C_0) \subsetneq e(A) \subset \mathcal{S}.$$

This implies that C_0, C_1,... are distinct formula-occurrences in $\mathcal{S}$, thus $C_1 \sqcup D_1$, $C_2 \sqcup D_2$,... are also distinct. But $\mathcal{S}$ is finite, thus we cannot have an infinite descending sequence of proper inclusions. Therefore there is an n such that D_n is the main door of $e(B)$, hence $B \in e(A)$. ∎

Tiling Lemma. (cfr. Girard [1987c] II.1, Proposition 3) *Let $\mathcal{S}$ be a proof-net for* **MLL**. *Let $A, B \in \mathcal{S}$ such that $e(A) \cap e(B) \neq \emptyset$. If $A \notin e(B)$, then $B \sqsubset A$.*

Proof. We show that under the given conditions, $A \notin e(B)$ implies $B \in e(A)$. Fix $X \in e(A) \cap e(B)$, consider any normal computation of $X \in e(A)$ and let $\tau^{X}_{[A]}$ be the corresponding tree of admissible empire-sequences. Since $A \notin e(B)$, any empire-sequence in $\tau^{X}_{[A]}$ must enter $e(B)$ through some door D by a step of the form $(\uparrow)$. The result follows by taking the subtree $\tau^{D}_{[A]}$ and applying the previous proposition (in the direction $(ii) \Rightarrow (i)$. ∎

Door Lemma. *Let $\mathcal{S}$ be a proof-net for* **MLL** *and let D be any door of $e(A)$, for some $A \in \mathcal{S}$. Then $A \Diamond D$, and $e(A) \subset e(D)$. In particular, if D and D' are any two doors of $e(A)$, $D \Diamond D'$.*

Proof. Suppose $A \notin e(D)$. Since D is a door of $e(A)$, we have $D \sqsubset A$. By the previous Proposition, for some normal computation of $D \in e(A)$ the last step in

$\tau^D_{[A]}$ has the form ($\uparrow$). But this is impossible, since D is a door of $e(A)$. Therefore $D\Diamond A$.

Suppose $X \in e(A)$ and let $\tau^X_{[A]}$ be a normal tree. No step $\Downarrow\frac{D \quad E}{F}$ can occur in an empire-sequence of $\tau^X_{[A]}$, since D is a door of A. Taking any tree $\tau^A_{[D]}$ and applying the Quasi-Transitivity Lemma, we obtain $X \in e(D)$, hence $e(A) \subset e(D)$.

Thus if D and D' are two doors of $e(A)$, we obtain $D\Diamond D'$ by applying twice the previous paragraph. ∎

Shared Empires Lemma. (cfr. Girard [1987c] II.2, Definition 3) *Let S be a proof-net for* MLL *and suppose $A\Diamond B$.*

(i) *If for some X, $X \in e(B)$ and $X \notin e(A)$, then any normal computation of $X \in e(B)$ contains a step of the form $\downarrow \frac{A}{E}$, for some E and all formulas labeling the subtree $\tau^A_{[B]}$ of $\tau^X_{[B]}$ belong to $e(A) \cap e(B)$* (i.e., any normal computation of $X \in e(B)$ must exit $e(A)$ first through the main door.)

(ii) *$e(A) \cap e(B)$ is the set of all formulas X such that for some normal computation of $X \in e(B)$, the nodes of $\tau^X_{[B]}$ are labeled with formulas of $e(A) \cap e(B)$ only.*

Proof. (i) Fix $X \in e(B)$, with $X \notin e(A)$ and consider a normal computation of $X \in e(B)$. Any empire-sequence in $\tau^X_{[B]}$ must exit $e(A)$ through a door D by a step $\downarrow \frac{D}{E}$, for some E. We claim that some door D is the main door A.

Otherwise, in any occurrence of such a step F must be have the form $C \sqcup D$ or $D \sqcup C$ and another empire-sequence in $\tau^X_{[B]}$ contains the step $\downarrow \frac{C}{E}$. But $C \notin e(A)$, thus any empire-sequence in the subtree $\tau^C_{[B]}$ must exit $e(A)$ through a door D' by a step $\downarrow \frac{D'}{E'}$, for some E'.

By repeating this argument, we find an infinite sequence $E, E', E'', \ldots$ which must be all distinct, since $\tau^X_{[B]}$ is normal. But $\tau^X_{[B]}$ is finite, a contradiction.

Finally, consider a subtree $\tau^A_{[B]}$ of $\tau^X_{[B]}$. If some node in $\tau^A_{[B]}$ is labeled by a $Y \notin e(A)$, then by the previous argument some empire-sequence in $\tau^A_{[B]}$ must exit $e(A)$ through the main door. But $\tau^A_{[B]}$ is normal, since $\tau^X_{[B]}$ is normal, a contradiction.

(ii) For $X \in e(A) \cap e(B)$, consider a normal computation of $X \in e(B)$ represented by the normal tree $\tau^X_{[B]}$. If $e(B) \subset e(A)$, then all the labels of $\tau^X_{[B]}$ are formula-occurrences in $e(A)$, as required. Otherwise, suppose some node in $\tau^X_{[B]}$ is labeled with $Y \notin e(A)$. By part (i) an empire-sequence in $\tau^X_{[B]}$ contains a step of the form $\downarrow \frac{A}{E}$, for some E. Since $X \in e(A)$, the same empire sequence must eventually reenter

$e(A)$ through a step of the form $\uparrow \frac{D'}{E'}$. Since $\tau^X_{[B]}$ is normal, so is the subtree $\tau^{D'}_{[B]}$. By the above Proposition, $A \sqsubset B$, contradicting the assumption $A \Diamond B$. ■

Substructure Theorem. *Let $\mathcal{S}$ be a proof-net for* MLL. *For all $A \in \mathcal{S}$, define $\mathcal{S}_A(\Delta) =_{df} e(A)$ where Δ contains exactly the doors of $e(A)$ and let $\overline{\mathcal{S}}^A$ be the complementary substructure of $\mathcal{S}_A$* (see section (3.3): this is a structure with nonlogical axiom $\overline{\Delta}$). *Then*

(i) *for all $X, Y \in \mathcal{S}_A$, $Y \in e(X)$ in $\mathcal{S}_A$ if and only if $Y \in e(X)$ in $\mathcal{S}$.*

(ii) *for all $X, Y \in \overline{\mathcal{S}}^A$, $Y \in e(X)$ in $\overline{\mathcal{S}}^A$ if and only if $Y \in e(X)$ in $\mathcal{S}$.*

(iii) *$\mathcal{S}_A$ and $\overline{\mathcal{S}}^A$ satisfy the vicious circle and connectedness conditions, i.e., they are* sub nets.

Proof. (*i*) Clearly, if $Y \in e(X)$ in $\mathcal{S}_A$, then $Y \in e(X)$ in $\mathcal{S}$. Conversely, if $Y \in e(X)$ in $\mathcal{S}$, then by the Shared Empires Lemma some computation of $Y \in e(X)$ remains always inside $e(A) = \mathcal{S}_A$.

(*ii*) Assume $Y \in e(X)$ in $\overline{\mathcal{S}}^A$ and consider a normal tree $\overline{\tau}^Y_{[X]}$ of admissible empire-sequences, labeled with formula-occurrences in $\overline{\mathcal{S}}^A$. By induction on $\overline{\tau}^Y_{[X]}$ we construct a tree $\tau^Y_{[X]}$ which is labeled with formula-occurrences in $\mathcal{S}$ and has the following property: for every step in $\overline{\tau}^Y_{[X]}$ from U to V different from

$$\underset{\longrightarrow}{\overline{\cdots U \cdots V \cdots}}$$

where U and V are in $\overline{\Delta}$, the nonlogical axiom of $\overline{\mathcal{S}}^A$, there is a step of the same kind in $\tau^Y_{[X]}$ from U to V. The construction is as follows. Every step in $\overline{\tau}^Y_{[X]}$ of the form $\underset{\longrightarrow}{\overline{\cdots U \cdots V \cdots}}$ where $U, V \in \overline{\Delta}$, follows a step of the form (*uparrow*), since $\overline{\tau}^Y_{[X]}$ is normal. By inductive hypothesis we have a $\tau^U_{[X]}$ ending with a step ($\uparrow$). By the Door Lemma, $V \in e(U)$ in $\mathcal{S}$. By the Shared Empires Lemma we can construct a normal tree $\tau^V_{[U]}$ labeled only with formula-occurrences in $e(A)$. Therefore in $\tau^V_{[U]}$ there is no step of the form ($\downarrow$) or ($\Downarrow$) through $\frac{X}{Z}$ for some Z. Therefore the result of replacing each bottom node (leaf) of $\tau^V_{[U]}$ with a copy of $\tau^U_{[X]}$ represents a computation of $V \in e(X)$. The other steps of the construction are obvious.

Conversely, consider $X, Y \in \overline{\mathcal{S}}^A$ and assume $Y \in e(X)$ in $\mathcal{S}$. Given a normal tree $\tau^Y_{[X]}$ of admissible empire-sequences, labeled with formula-occurrences in $\mathcal{S}$, by induction on $\tau^Y_{[X]}$ we construct a normal tree $\overline{\tau}^Y_{[X]}$ labeled with formula-occurrences

in $\overline{S}^A$. Let $U \in \overline{\Delta}$. Since $X, Y \in \overline{S}^A$, every empire-sequence in $\tau^Y_{[X]}$ containig a step of the form $(\uparrow)\frac{W}{U}$ must also contain a step of the form $(\downarrow)\frac{Z}{V}$, where $V \in \Delta$. If $\overline{\tau}^U_{[X]}$ is the tree given by induction hypothesis, we continue each empire-sequence in $\overline{\tau}^U_{[X]}$ with a step of the form $\overline{\cdots \underset{\longrightarrow}{U \cdots V} \cdots}$. The rest of the construction is obvious.

(iii) This follows from parts (i) and (ii). ∎

4.4. Sequentialization Theorem for Propositional MLL.

We give a proof of the *Sequentialization Theorem* for Propositional MLL:

Sequentialization Theorem. *If S is a proof-net with conclusion Γ for* MLL$^-$ *or* MLL *then we can find a derivation $\mathcal{D}$ of $\vdash \Gamma$ in the corresponding fragment such that the associated proof-net $S_{\mathcal{D}}$ is S.*

4.4.1. Proof of the Sequentialization Theorem.

Proof. Suppose S is a proof-structure with conclusion Γ in propositional MLL satisfying the vicious circle and connectedness conditions. We proceed by induction on the number of formulas in S. If Γ consists of an axiom link, then a derivation of Γ consists of a sequent axiom possibly followed by a sequence of applications of the $\bot$-rule. If Γ contains the conclusion of a $\sqcup$-link, then the induction hypothesis applies to the proof-structure S' obtained from S by removing only that link and we conclude with an application of the $\sqcup$-rule.

The difficult step is the case when Γ contains no conclusions of a $\sqcup$-link. We use the following argument by Girard [1987c], II.1, Remark 2. We need to find a $\otimes$-link with conclusion $A \otimes B \in \Gamma$ with the property that if S' results form S by removing only the link in question, then S' is partitioned into two substructures $S_1 = e(A)$ and $S_2 = e(B)$ both satisfying the vicious circle and connectedness conditions. If for some $A \otimes B$ all the doors of $e(A)$ and $e(B)$ are fake doors, i.e., elements of Γ, then all the conclusions of S different from $A \otimes B$ are partitioned by $e(A)$ and $e(B)$. Otherwise, let Δ be $\Gamma \setminus (e(A) \cup e(B))$. If any axiom connects $X \in \Delta$ with $Y \in e(A)$ [or $Y \in e(B)$], then X contains a door of $e(A)$, [of $e(B)$], which is impossible. If no axiom connects Δ with either $e(A)$ or $e(B)$, then $\Delta \cap e(A \otimes B) = \emptyset$. This contradicts the connectedness condition. Therefore it is enough to find $A \otimes B$ such that neither $e(A)$ nor $e(B)$ has any side door.

Let $A_0 \otimes B_0 \in \Gamma$ and suppose $e(B_0)$ has a side door D. Then $D \prec X \in \Gamma$, and by the hypothesis of the case, $X = A_1 \otimes B_1$, say $D \prec A_1$. Since $D \in e(B_0) \cap e(A_1)$ and $A_1 \notin e(B_0)$, we have $B_0 \in e(A_1)$ and $e(B_0) \subset e(A_1)$. Moreover, $A_0 \in e(A_1)$, since $A_0 \otimes B_0 \in e(A_1)$ and if $A_1 \in e(A_0)$, then $D \in e(A_0) \cap e(B_0)$ and by the Tiling Lemma the vicious circle condition is contradicted. Thus $e(A_0) \subset e(A_1)$. Since $e(A_1) \cap e(B_1) = \emptyset$, we obtain

$$e(A_0) \cup e(B_0) \subsetneq e(A_1) \subsetneq e(A_1) \cup e(B_1).$$

If $e(B_1)$ has a side door D_1, then we can find a formula-occurrence $A_2 \otimes B_2$ such that $D_1 \prec A_2$, and arguing as before we obtain

$$e(A_1) \cup e(B_1) \subsetneq e(A_2) \subsetneq e(A_2) \cup e(B_2),$$

and so on. Plainly, $A_0 \otimes B_0$, $A_1 \otimes B_1$,... are different elements of Γ, thus the search ends with a $A_n \otimes P_n$ with the property that $e(A_n) \cup e(B_n)$ is maximal with respect to inclusion. Thus neither $e(A_n)$ nor $e(B_n)$ can contain any side door. ∎

4.5. Sequentialization Theorem for Propositional MALL.

The results of sections (4.3) and (4.4) are easily generalized to proof-nets for **MALL** once the following fact is available.

Box-Door Lemma. *Let S_φ be a proof-net for* **MALL** *and let D be any door of $e(A_i)$, where*

$$\frac{A_1 \quad A_2}{A_1 \sqcap A_2}$$

is a link in S_φ. Then $A_i \Diamond D$.

A proof of the Box-Door Lemma is given at the end of the section.

Fact. *Let S_φ be a proof-net for* **MALL**. *If*

$$\frac{X_1 \quad X_2}{X_1 \sqcap X_2}$$

occurs in S_φ and for some B and D, D is a door simultaneously of $e(B)$ and of $e(X_i)$, then $B \in e(D)$.

Proof. By the box condition, D is either X_i or F_j in a Contraction link

$$\frac{F_1 \quad F_2}{F}$$

where F_j is flagged with X_i, $i, j \leq 2$; say D is F_1 and F_i is flagged with X_i.

Suppose $Z \in e(X_i)$ and let $\tau^Z_{[X_i]}$ be a normal tree. No step $\Downarrow\frac{F_1 \quad F_2}{F}$ can occur in an empire-sequence of $\tau^Z_{[X_i]}$, since F_i is a door of X_i. By the Box-Door Lemma, $X_i \Diamond F_i$; taking any tree $\tau^{X_i}_{[F_i]}$ and applying the Quasi-Transitivity Lemma, we obtain $Z \in e(F_i)$. Hence $e(X_i) \subset e(F_i)$ and by part (i) of the Box Lemma (section 4.2) $e(X_i) = e(F_j)$.

Suppose $B \notin e(D)$, and consider a normal $\tau^D_{[B]}$. Any empire-sequence in $\tau^D_{[B]}$ reaches a door G_i of $e(D)$ for the first time by a step($\uparrow$) through a link

$$\frac{G_1 \quad G_2}{G},$$

where G_i is flagged with X_j, say $i = j$. By the Box-Door Lemma, $G_i \Diamond F_i$. By part (iii) of the Proposition in section (4.2), $F_1 \in e(B)$ and $F_2 \in e(B)$, therefore F_1 cannot be a door of $e(B)$, a contradiction, since F_1 is D. ∎

Proposition. *Let S_φ be a proof-net for* **MALL**. *The following are equivalent:*

(i) $B \sqsubset A$;

(ii) *for some door D of $e(B)$ and some normal computation of $D \in e(A)$, the last step in $\tau^D_{[A]}$ has the form $(\uparrow)$;*

(iii) *either some link*

$$\frac{X_1 \quad X_2}{X_1 \sqcap X_2}$$

occurs in $e(A)$ and B and X_i have a common door, or for some normal computation of $B \in e(A)$ the last step in $\tau^B_{[A]}$ has the form $(\uparrow)$.

Proof. As in the proof of the analogue Proposition in section (4.3), $(i) \Rightarrow (ii)$ is straightforward. To prove $(iii) \Rightarrow (i)$, let D' be a common door of $e(B)$ and $e(X_i)$, where

$$\frac{D' \quad D''}{D}$$

is a link in $e(A)$. By the above fact, $B \in e(D')$, thus $e(B) \subset e(X_i)$ by two applications of part (ii) of the Box Lemma, section (4.2). By parts (i) and (ii) of the

same lemma, $A \notin e(X_1)$ and $A \notin e(X_2)$, since $X_1, X_2 \in e(A)$. Therefore $A \notin e(B)$. Furthermore, for the same reason no empire-sequence in a tree $\tau^B_{[X_i]}$ can cross a link $\frac{A \quad W}{Z}$ for any W, Z. Combining $\tau^B_{[X_i]}$ with a tree $\tau^{X_i}_{[A]}$, we obtain $B \in e(A)$ by the Quasi-Transitivity Lemma. Hence $B \sqsubset A$. The other case of $(iii) \Rightarrow (i)$ is as before.

$(ii) \Rightarrow (iii)$. Let D be a door of $e(B)$ and let $\tau^D_{[A]}$ be a normal tree ending with a step $(\uparrow)$ through a link

$$\frac{C \quad D}{E}.$$

If D is B then we are done. If D is a common door of $e(B)$ and of X_i, say of X_1, for some link

$$\frac{X_1 \quad X_2}{X_1 \sqcap X_2},$$

then by the Box-Door Lemma, $D \Diamond X_1$ and $C \Diamond X_2$. By part (iii) of the Proposition in Section (4.2), $e(C) \subset e(A)$ and $e(D) \subset e(A)$, therefore $X_1, X_2 \in e(A)$ and so $X_1 \sqcap X_2 \in e(A)$, as required.

If the link $\frac{C \quad D}{E}$ is a $\sqcup$-link, then we consider a normal tree $\tau^D_{[C]}$ and we proceed as in the proof of direction $(ii) \Rightarrow (iii)$ of the Proposition in section (4.3). We construct a sequence D_0, D_1,... of doors of $e(B)$ and a sequence C_0, C_1,... such that

$$\ldots \subsetneq e(C_1) \subsetneq e(C_0) \subsetneq e(A) \subset \mathcal{S}.$$

In a finite number of steps we reach either the main door B or a common door of $e(B)$ and of $e(X_i)$, for some premise X_i of a $\sqcap$-link. ∎

Tiling Lemma. *Let $\mathcal{S}_\varphi$ be a proof-net for* **MALL**. *Let $A, B \in \mathcal{S}$ such that $e(A) \cap e(B) \neq \emptyset$. If $A \notin e(B)$, then $B \sqsubset A$.*

Proof. As in the case of **MLL**, the proof follows by an application of the above proposition in the direction $(ii) \Rightarrow (i)$. ∎

Door Lemma. *Let $\mathcal{S}_\varphi$ be a proof-net for* **MALL** *and let D be any door of $e(A)$, for some $A \in \mathcal{S}_\varphi$. Then $A \Diamond D$, and $e(A) \subset e(D)$.*

Proof. If $A \notin e(D)$, then by the above Fact D cannot be simultaneously a door of A and of X_i, for some link

$$\frac{X_1 \quad X_2}{X_1 \sqcap X_2}.$$

Also, no $\tau^D_{[A]}$ can end with a step $(\uparrow)$, since D is a door of A. As before, we reach a contradiction by applying the above Proposition in the direction $(i) \Rightarrow (iii)$. ∎

The Shared Empires Lemma carries through in **MALL**, in fact it is trivial if applied to premises of $\sqcap$- or Contraction links, since in this case $A \Diamond B$ implies $e(A) = e(B)$, by the Box Lemma (section 4.2). The Substructure Theorem also follows in **MALL** from the Box-Door Lemma.

It would be reasonable to regard the Box-Door Lemma as a basic connectedness condition in the definition of proof-nets for **MALL**. However, it can be derived by a straightforward induction on $\triangleleft$.

Proof of the Box-Door Lemma. Let $\mathcal{S}_\varphi$ be a proof-net for **MALL**, let

$$\frac{A_1 \quad A_2}{A_1 \sqcap A_2}$$

be a link in $\mathcal{S}_\varphi$ and D is any door of $e(A_i)$. If no $\sqcap$-link occurs in $e(A_i)$, then $D \Diamond A$ by the Door Lemma for **MLL** (easily extended to the fragment with $\oplus$). Otherwise, it follows from the Door Lemma for **MALL** with the Box-Door Lemma inductively applied to the $\sqcap$-links inside $e(A_i)$. ∎

Sequentialization Theorem. *If $\mathcal{S}_\varphi$ is a proof-net with conclusion Γ for* **MALL** *then we can find a derivation $\mathcal{D}$ of $\vdash \Gamma$ in* **MALL** *such that the associated proof-net $\mathcal{S}_\mathcal{D}$ is $\mathcal{S}$.*

Proof. The argument is similar to the case for **MLL**. In the difficult step, when Γ contains no conclusions of a $\sqcup$-link, if every formula-occurrence in Γ is the conclusion of a $\sqcap$- or Contraction link, then all the premises of such links must be flagged with the premises of a unique link

$$\frac{A_1 \quad A_2}{A_1 \sqcap A_2},$$

by connectedness, and we proceed in the obvious way. Otherwise, we reduce to the argument in the proof for **MLL** (with nonlogical axioms) by repeated applications of the Substructure Theorem. The details are omitted. ∎

4.6. Sequentialization Theorem for First Order MALL.

The following proposition guarantees that the relation $<_t$ is well-behaved with respect to substructures. This is the only technical fact needed to extend tha Sequentialization Theorem for **MALL** to the first order case.

Proposition. *Let $\mathcal{S}$ be a proof-net for first order* **MALL**. *Let a, b be eigenvariables, where a is associated with $\bigwedge x.A(x)$. If $b = a$ or $b <_t a$, then b occurs only inside $e(A(a))$.*

Proof. Let $a = a_0 >^0 \ldots >^0 a_n = b$, where a_i is the eigenvariable associated with $\bigwedge x_i.A_i$. The proof is by induction on n. If $b = a$, then a occurs only in $e(A)$, since $<_t$ is strict.

If $a >_t b$, then since a_i occurs inside $e(A_{i+1})$ and a_i occurs only inside $e(A_i)$, for some formula $X(a_i)$, we have $X(a_i) \in e(A_i) \cap e(A_{i+1})$ and since by induction hypothesis a_i occurs only inside $e(A_0)$, $X(a_i) \in e(A_0) \cap e(A_{i+1})$. Since a_i occurs outside $e(A_{i+1})$, we cannot have $e(A_0) \subset e(A_{i+1})$, in particular $A_0 \sqsubset A_{i+1}$ and $A_0 = A_{i+1}$ are impossible. By the Tiling Lemma $A_{i+1} \sqsubset A_0$ or $A_{i+1} \Diamond A_0$. In both cases, a_{i+1} occurs inside $e(A_0)$; if a_{i+1} occurs also outside $e(A_0)$, $a_{i+1} >^0 a_0$, contradicting he fact that $<_t$ is strict.

Proof of the Sequentialization Theorem. Let $\mathcal{S}(\Gamma)$ be a proof-structure for first order **MALL** with flagging function φ, satisfying the vicious circle, connectedness, box and parameters conditions. By induction on $>_t$ we reduce to the proof for propositional **MALL**. Let $\bigwedge x.A$ have eigenvariable a, where a is maximal with respect to $<_t$. Let $\mathcal{S}_{\wedge x.A} =_{df} e(\bigwedge x.A)$ and let $\overline{\mathcal{S}}^{\wedge x.A}$ be the substructure of $\mathcal{S}$ complementary to $\mathcal{S}_{\wedge x.A}$ (containing the nonlogical axiom $\overline{\Delta, \bigwedge x.A}$). Since no ghost formula is a premise of any link, we may assume that Δ does not contain any ghost formula. Also let $\mathcal{S}_A(\Delta, A(a))$ be the substructure of $\mathcal{S}_{\wedge x.A}$ coinciding with $e(A(a))$.

Let $<_t^a$ be the relation $<_t$ defined on $\mathcal{S}_A$. If $b <_t^a c <^0 b$ in $\mathcal{S}_A$, then by the above Proposition, $b <_t c <^0 b$ in $\mathcal{S}$. Thus $<_t^a$ is a strict ordering. Notice that no eigenvariable c can occur in any fake door of $e(A(a))$, since no eigenvariable occurs in the conclusions of $\mathcal{S}$, nor in any side door of $e(A(a))$, since by the choice of a we cannot have $c >^0 a$.

By the Substructure Theorem, if X and Y occur in $\overline{\mathcal{S}}^{\wedge x.A}$, then $X \in e(Y)$ in $\overline{\mathcal{S}}^{\wedge x.A}$ if and only if $X \in e(Y)$ in $\mathcal{S}$. It follows that if $b <_t c <^0 b$ in $\overline{\mathcal{S}}^{\wedge x.A}$ then $b <_t c <^0 b$ in $\mathcal{S}$. We conclude that $\overline{\mathcal{S}}^{\wedge x.A}$ satisfies the parameters condition.

By the Substructure Theorem (Corollary, section 4.5) and the above facts, $\mathcal{S}_{\wedge x.A}$ and $\overline{\mathcal{S}}^{\wedge x.A}$. are subnets. Therefore we can apply the induction hypothesis to $\mathcal{S}_A(\Delta, A(a))$ and to $\overline{\mathcal{S}}^{\wedge x.A}$ and obtain a derivation $\mathcal{D}_0$ with conclusion $\vdash \Delta, A(a)$ and a derivation $\mathcal{D}_1$ of $\vdash \Gamma$ with *non-logical sequent-axiom* $\vdash \Delta, \bigwedge x.A$.

As pointed out before, a does not occur in Δ, therefore

$$\frac{\vdash \Delta, A(a)}{\vdash \Delta, \bigwedge x.A}$$

is a correct inference. Therefore the concatenation of $\mathcal{D}_0$ and $\mathcal{D}_1$ is a correct derivation of Γ. ∎

4.7. A System of Annotations on Proof-Structures.

We consider in detail a notational variant of Proof-Structures inspired by the *sequent formulation* of Natural Deduction and described in section (1.1.5).

Suppose we annotate each formula A in a proof-structure $\mathcal{S}$ with

$$(\Gamma)[\Delta] - A,$$

where Γ is the list of *open* doors and Δ is the list of *closed* doors of the empire of A (*fake* and *side* doors in Girard's terminology). In the case of a *first order proof-structure* if a formula $\bigwedge x.A$ is associated with the eigenvariable a, then we add $a <^0 \{b_1 \dots b_n\}$ to the annotations of $\bigwedge x.A$, where $a <^0 b_i$ for all $i \leq n$. In a LISP environment, such annotations may be simply pointers to other parts of the structure. Notice that in a given proof-structure the empire of a formula is completely determined by its doors.

First we consider the properties of the annotations when $\mathcal{S}$ is a proof-net.

(1) Let $\mathcal{S}$ be a proof-net ending with a link

$$\frac{B \quad C}{B \otimes C}$$

and let $\mathcal{S}_1$ be the result of removing the link in question. Suppose A occurs in $\mathcal{S}_1$ with the annotation $(\Gamma)[\Delta] - A$.

(i) If Γ is Γ', B, the annotation of C in $\mathcal{S}_1$ is $(\Pi), [\Lambda] - C$ and $\underline{B \otimes C}$ is the ghost conclusion of cut (thus $C = B^\perp$), then the annotation of A in $\mathcal{S}$ is $(\Gamma', \Pi)[\Delta, \Lambda] - A$;

(ii) If Γ is Γ', B, the annotation of C in $\mathcal{S}_1$ is $(\Pi), [\Lambda] - C$ and $B \otimes C$ is not a cut, then the annotation of A in $\mathcal{S}$ is $(\Gamma', B \otimes C, \Pi)[\Delta, \Lambda] - A$.

Similar clauses hold if $\Gamma = \Gamma', C$.

(iii) If B, C do not occur in Γ, then the annotation of A in $\mathcal{S}$ is still $(\Gamma)[\Delta] - A$.

This is precisely what we expect if we regard the annotations (1) Let $\mathcal{S}$ be a proof-net ending with a link

$$\frac{B \qquad C}{B \otimes C}$$

and let $\mathcal{S}_1$ be the result of removing the link in question. Suppose A occurs in $\mathcal{S}_1$ with the annotation $(\Gamma)[\Delta] - A$.

(i) If Γ is Γ', B, the annotation of C in $\mathcal{S}_1$ is $(\Pi), [\Lambda] - C$ and $B \otimes C$ is a cut (thus $C = B^\perp$), then the annotation of A in $\mathcal{S}$ is $(\Gamma', \Pi)[\Delta, \Lambda] - A$;

(ii) If Γ is Γ', B, the annotation of C in $\mathcal{S}_1$ is $(\Pi), [\Lambda] - C$ and $B \otimes C$ is not a cut, then the annotation of A in $\mathcal{S}$ is $(\Gamma', B \otimes C, \Pi)[\Delta, \Lambda] - A$.

Similar clauses hold if $\Gamma = \Gamma', C$.

(iii) If B, C do not occur in Γ, then the annotation of A in $\mathcal{S}$ is still $(\Gamma)[\Delta] - A$.

This is precisely what we expect if we regard the annotations as the analogue of a list of the open and closed assumptions in ordinary Natural Deduction and if we think of the introduction of a $\otimes$-link as the analogue of the substitution of a derivation for an open assumption of another derivation.

(2) Let $\mathcal{S}$ be a proof-net ending with a link

$$\frac{B \quad C}{B \sqcup C}$$

and let $\mathcal{S}_1$ be the result of removing the link in question. Suppose A occurs in $\mathcal{S}_1$ with annotation $(\Gamma)[\Delta] - A$.

(i) If $\Gamma = \Gamma', B, C$, then the annotation of A in $\mathcal{S}$ is $(\Gamma', B \sqcup C)[\Delta] - A$;

(ii) If $\Gamma = \Gamma', B$, where C does not occur in Γ', then the annotation of A in $\mathcal{S}$ is $(\Gamma')[\Delta, B] - A$;

(iii) If B and C do not occur in Γ, then the annotation of A in $\mathcal{S}$ is still $(\Gamma)[\Delta] - A$.

The second case corresponds precisely to the removal of open assumptions in the implication introduction rule.

Finally, let A have the form $\bigwedge x.A'$ and let $a <^0 \{b_1, \dots, b_k\}$ be also in the annotation of A in $\mathcal{S}_1$. In addition, suppose we are in case (ii) above, so that a door of $e(A)$, say B, becomes closed in $\mathcal{S}$, and that b is an eigenvariable occurring in B with $b \neq b_i$ for all $i \leq k$. Then $a <^0 \{b_1, \dots, b_n, b\}$ is in the annotation of A in $\mathcal{S}$.

(3) Let $\mathcal{S}$ be a proof-net ending with links

$$\cdots\frac{C_i' \quad C_i''}{C_i}\cdots \qquad \frac{B \qquad C}{B \sqcap C}$$

where the annotations of B and C are $[C_1', ..., C_n'] - B$ and $[C_1'', ..., C_n''] - C$. Let $\mathcal{S}_1$ be the result of removing all the links in question. If A occurs inside $\mathcal{S}_1$ with the annotation $(C_{i_1}', ..., C_{i_p}')[\Lambda] - A$ or $(C_{i_1}'', ..., C_{i_p}'')[\Lambda] - A$, then the annotation of A in $\mathcal{S}$ is $[C_{i_1}', ..., C_{i_p}', \Lambda] - A$ or $[C_{i_1}'', ..., C_{i_p}'', \Lambda] - A$. Here $0 \leq p \leq n$ and these are the only possible cases, since $\mathcal{S}$ satisfies the box condition.

Let A occur in $\mathcal{S}_1$ and have the form $\bigwedge x.A'$ and let

$$a <^0 \{b_1, \dots, b_k\}(C_{i_1}', ..., C_{i_p}')[\Lambda] - A$$

be the annotation of A in $\mathcal{S}_1$. In addition, suppose that $c_1,...,c_q$ are all the eigenvariables occurring in either $C_{i_1}',...,C_{i_p}'$ or in B with $c_l \neq b_j$ for all $j \leq k, l \leq p$. Then let

$$a <^0 \{b_1, \dots, b_k, c_1, \dots, c_q\}[C_{i_1}', \dots, C_{i_p}', \Lambda] - A$$

be the annotation of A in $\mathcal{S}$.

(4) Let $\mathcal{S}$ be a proof-net ending with the link

$$\frac{A(a)}{\bigwedge x.A(x)}$$

and let $\mathcal{S}_1$ be the result of removing the link in question from $\mathcal{S}$. Let $\bigwedge y.B$ occur in $\mathcal{S}_1$ with the associate eigenvariable b and with the annotation

$$b <^0 \{d_1, \ldots, d_m\}(\Gamma)[\Delta] - \bigwedge y.B.$$

If a occurs in Δ or in $\bigwedge y.B$ then we add a to the list $\{d_1, \ldots, d_m\}$ in the annotation of $\bigwedge y.B$ in $\mathcal{S}$.

Conversely, the annotations can be used to verify whether a proof-structure is a proof-net. For instance, in the case of a link $\frac{B \quad C}{B \otimes C}$, to verify the vicious circle condition we consider the annotations of B and C, say $(\Pi)[\Lambda] - B$ and $(\Xi)[\Theta] - C$, and check that no formula-occurrence in $\Pi \cup \Lambda$ is a subformula of $\Xi \cup \Theta$.

In the case of a link $\frac{B \quad C}{B \sqcup C}$, to verify the connectedness condition we consider the annotations of B and C, say $(\Pi)[\Lambda] - B$ and $(\Xi)[\Theta] - C$, and check that Π contains the same occurrences as Ξ and Λ the same occurrences as Θ.

In the case of a link $\frac{B \quad C}{B \sqcap C}$, to verify the box condition we consider the annotations of B and C, say $(C_1, ..., C_n) - B$ and $(C'_1, ..., C'_n) - C$ and verify that for all $i \leq n$ there is a contraction link $\frac{C_i \quad C'_i}{C''_i}$.

Finally, we use all the annotations of the form $a <^0 \{b_1, \ldots, b_k\}$ to check that we do not obtain $c <_t c$ for some c, etc.

5. The Method of Chains.

The notions of *path* and of *chain* (section 5.1) are the main data-structure in Ketonen and Weyhrauch [1984] and allow an alternative treatment of proof-nets for **MLL**. Sections (5.2, 5.3) solve the problem of comparing the *method of chains* and the *method of empires* for **MLL**.

5.1. Chains.

In a proof-structure $\mathcal{S}$ with conclusion Γ a path is simply a set of axiom links satisfying certain conditions and a chain is a global connection, determined by the axiom-links and the $\otimes$-links.

Chain

$$
\begin{array}{cccccc}
\multicolumn{2}{c}{\overline{\hspace{4em}}} & \overline{\hspace{2em}} \cdots & \overline{\hspace{2em}} & \multicolumn{2}{c}{\overline{\hspace{4em}}} \\
P_1 & P_1^{\perp} & P_2 & P_{n-1}^{\perp} & P_n & P_n^{\perp} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
A & \multicolumn{2}{c}{C_1 \otimes C_2} & \multicolumn{2}{c}{C_{n-1} \otimes C_n} & B\ .
\end{array}
$$

In a chain, *the structure of the subformulas lying between axiom-links and $\otimes$-links is ignored*: for this reason, chains have simple properties. For instance, if C is a chain from A to B, then C is also a chain from B to A (*symmetry*); if C_1 is a chain from A to B, C_2 is a chain from C to D, then $A - C_1 - B \otimes C - C_2 - D$ is a chain from A to D.

Definitions. Let $\mathcal{S}(\Gamma)$ be a proof-structure in **MLL**.

(i) A *path* $\mathcal{P}$ in $\mathcal{S}$ is a nonempty set of axiom links in $\mathcal{S}$ satisfying the *relevance condition on conjunctions* (defined below).
If A is a formula occurrence of $\mathcal{S}$, we write $\mathcal{P} \mapsto A$ (A *is relevant relatively to* $\mathcal{P}$) if for some axiom $\overline{X_1, \ldots, X_n} \in \mathcal{P}$ and some $i \leq n$, $X_i \prec A$.
The *relevance condition on conjunctions* is the property

for every $A_1 \otimes A_2$, *if* $\mathcal{P} \mapsto A_1 \otimes A_2$ *then* $\mathcal{P} \mapsto A_1$ *and* $\mathcal{P} \mapsto A_2$.

(ii) A path $\mathcal{P}$ in $\mathcal{S}$ is minimal if no proper nonempty subset $\mathcal{P}'$ of $\mathcal{P}$ satisfies the relevance condition on conjunctions.

(iii) For distinct $A, B \in \mathcal{S}$, we say that A and B are *connected* (write $A \parallel B$) if there is an axiom $\overline{X_1, \ldots, X_n} \in \mathcal{P}$ such that $X_i \prec A$ and $X_j \prec B$ with $i \neq j$.

(iv) Let C and D be formula-occurrences in $\mathcal{S}$ such that $D \not\preceq C$ and $C \not\preceq D$ and suppose $A \prec C$ and $B \prec D$. We write $C_A -_B D$ if $A \parallel B$.

Combining (iv) and (v) we obtain the ternary relation $\mathcal{R}(Y, X, Z)$ of Ketonen and Weyhrauch [1984], definition 4.3. $\mathcal{R}(Y, X, Z)$ holds if and only if for some A, B, we have $Y -_A X_B - Z$.

(v) We write ${}_A C_B$ if $A \otimes B \preceq C$ in $\mathcal{S}$.

(vi) For $n > 0$ a *chain* of length n is a set $\mathcal{C} = C_0, \ldots, C_n$ of formula-occurrences in $\mathcal{S}$ such that ${}_{A_1}C_{1B_1} - \ldots -_{A_n} C_{nB_n}$, for some $A_1, \ldots, A_n$, $B_1, \ldots, B_n$.

(vii) A chain is *pure* if for all $i \neq j$ with $1 \leq i, j \leq n$, $C_i \not\preceq C_j$.

$(viii)$ A *cycle of length* 1 is a formula ${}_A C_B$ such that $A \parallel B$. A *cycle of length* n, for $n > 1$, is a chain

$$ {}_A C_{0B} -_{A_1} C_{1B_1} - \ldots -_{A_{n-1}} C_{n-1B_{n-1}} -_A C_{0B} $$

where $C_0, \ldots, C_{n-1}$ is pure.

For instance, the path $\overline{P^\perp, P}$, $\overline{Q, Q^\perp}$, $\overline{R, R^\perp}$ and $\overline{S, S^\perp}$ determines a cycle in

$$ P^\perp \otimes Q, Q^\perp \otimes R, R^\perp \otimes S, S^\perp \otimes P, $$

and the path $\overline{P^\perp, P}$, $\overline{Q, Q^\perp}$ and $\overline{R, R^\perp}$ determines a chain but not a cycle in

$$ P^\perp \otimes Q, Q^\perp \otimes R, R^\perp, P. $$

The notions of chains and cycles are fundamental in our applications.

Given a proof-structure $\mathcal{S}$ in **MLL**, a formula $A \in \mathcal{S}$ and a path $\mathcal{P}$ in $\mathcal{S}$, we define $e_{\mathcal{P}}(A)$, the empire of A *relative to* $\mathcal{P}$, in the same way as $e(A)$ in section (3.4), except that condition (v) is now replaced with

$(v)'$ for any $\sqcup$-link in $\mathcal{S}$, if all the premises X of the link *such that* $\mathcal{P} \mapsto X$ are different from A and all belong to $e_{\mathcal{P}}(A)$, then the conclusion belongs to $e_{\mathcal{P}}(A)$.

5.2. No Vicious Circles versus No Cycles.

Can we determine the structure of inference in an equivalent way by the method of chains and of empires?

The answer is: *chains* allow an analysis of deduction in a more general context than empires.

In order to make the answer precise, *suppose the connectednss condition is satisfied* by a proof-structure $\mathcal{S}$. If

$$\mathcal{C}: \qquad A_0 \otimes B_0 - A_1 \otimes B_1 - \ldots - A_n \otimes B_n - A_0 \otimes B_0$$

is a chain where for $0 \leq i \neq j \leq n$, $A_i \otimes B_i \not\prec A_j \otimes B_j$, then by the Proposition below there is an $i \leq n$ such that $\mathcal{C} \subset e(A_i) \cup e(B_i)$. It is easy to see that in this case the vicious circle condition is violated.

Conversely, if the vicious circle condition is violated in $\mathcal{S}$, consider a substructure $\mathcal{S}'$ of $\mathcal{S}$, such that there Δ contains a the conclusion of a link $\frac{A \quad B}{A \otimes B}$ with the following property. If $\mathcal{S}_0$ is $e(A) \cup e(B)$ in $\mathcal{S}'$, then the vicious circle and connectedness condition are satisfied in $\mathcal{S}_0$ and $e(A) \cap e(B) \neq \emptyset$. Such $\mathcal{S}'$ can always be obtained, by deleting links of $\mathcal{S}$ in a suitable way. By the Tiling Lemma (section 4.3), $A \in e(B)$ or $B \in e(A)$. By the previous paragraph there is no cycle in $\mathcal{S}_0$. By the Lemma below there is a chain

$$\mathcal{C}: \qquad B - A_1 \otimes B_1 - \ldots - A_n \otimes B_n - A$$

where if we let $A_0 \otimes B_0$ be $A \otimes B$, then for $0 \leq i \neq j \leq n$, $A_i \otimes B_i \not\prec A_j \otimes B_j$. It follows that $\mathcal{S}$ contains a cycle.

Remark. Given a proof-structure $\mathcal{S}$ in **MLL**, the results below in this section together with the the above argument prove that if the connectedness conditions holds then the vicious circle condition is equivalent to the condition *there is no cycle.* On the other hand, it will be shown in section (8.3) that the condition *there is no cycle* is a consistency requirement that applies to more general situations, when the connectedness condition fails.

Proposition *Let $\mathcal{S}$ be a proof-structure for* **MLL** *satisfying the vicious circle and connectedness conditions. Let $\mathcal{P}$ be the set of axiom links. If $C = C_0 \otimes D_0 - \ldots - C_n \otimes D_n$ is a pure chain, then there is an $i \leq n$ such that $C \subset e(C_i) \cup e(D_i)$.*

Proof. Suppose $C \not\subset e(C_0) \cap e(D_0)$. Consider the first $j > 0$ such that $C_j \notin e(D_0)$. Notice that $D_{j-1} \in e(D_0)$. This is trivial, if $j - 1 = 0$; if $j - 1 \neq 0$, $D_{j-1} \notin e(D_0)$ would imply $C_{j-1} \notin e(D_0)$. Therefore there must be a door D of $e(D_0)$ such that $D \prec C_j$. Since $e(D_0) \cap e(C_j) \neq \emptyset$ and $C_j \notin e(D_0)$, by the Tiling Lemma we must have $e(D_0) \subsetneq e(C_j)$. Furthermore, by the definition of empire this fact implies $C_0 \in e(C_j)$.

If $C_j \in e(C_0)$, then $D \in e(C_0) \cap e(D_0)$, contradicting the vicious circle condition. Therefore $C_j \notin e(C_0)$ and by the Tiling Lemma we must have $e(C_0) \subsetneq e(C_j)$. Hence

$$e(C_0) \cup e(D_0) \subsetneq e(C_j) \cup e(D_j).$$

Now we repeat the argument and find $C_{j'} \otimes D_{j'}$ with

$$e(C_j) \cup e(D_j) \subsetneq e(C_{j'}) \cup e(D_{j'}),$$

and so on. Since at each step a different $C_k \otimes D_k$ is considered, after at most $n+1$ steps we find a $C_i \otimes D_i$ satisfying the Proposition. ■

Lemma. *Let $\mathcal{S}$ be a proof-structure for* **MLL** *let $\mathcal{P}$ be the set of axiom links and suppose there is no cycle. For all A, B in $\mathcal{S}$, if $B \in e(A)$ and $B \not\prec A$, then there is a pure chain $A' - C - B'$, where $A \preceq A'$ and $B \preceq B'$.*

Proof. Let $\tau^B_{[A]}$ represent a normal computation of $B \in e(A)$. C is constructed by selecting formula-occurrences in the nodes of some suitable empire-sequence of $\tau^B_{[A]}$. Starting from B, we scan the steps of an empire-sequence of $\tau^B_{[A]}$. We proceed in stages and in each stage we proceed alternatively "upwards" and a "downwards", according to the form of the steps in the empire-sequence. If the last step in $\tau^B_{[A]}$ is of the form $(\uparrow)$, then we are in a "upward" part of the empire-sequence and we let B' be the lowermost formula-occurrence of $\mathcal{S}$ considered at that stage. By the construction below, this is a conjunction $C = D_1 \otimes D_2$. Otherwise, we let $B' = B$ and we are in the "downward" part of the empire-sequence.

Stage p, *Downward Part*: we have a normal tree $\tau^X_{[A]}$ ending with a step of the form $(\downarrow)$ or $(\Downarrow)$. Starting from X, by the part (ii) of the Proposition in Section (4.1.1) we find a formula-occurrence X_j belonging to an axiom link $\overline{\cdots X_i \cdots X_j \cdots}$ such that $X_j \prec X$ and after a step $(\overrightarrow{i,j})$ from the node X_i to the node X_j the branch ends with a sequence of steps of the kind $(\downarrow)$ or $(\Downarrow)$. Let $\tau^{X_i}_{[A]}$ be the subtree of $\tau^X_{[A]}$ ending with X_i. Since $\tau^X_{[A]}$ is normal, so is $\tau^{X_i}_{[A]}$ and moreover the last step of $\tau^{X_i}_{[A]}$ has the form $(\uparrow)$. Next, we consider $\tau^{X_i}_{[A]}$ and proceed through the "upward" part of the empire-sequence.

Stage p, *Upward Part*: we have a normal tree $\tau^X_{[A]}$ ending with a step of the form $(\uparrow)$. By part (i) of the Proposition in Section (4.1.1), either $X \preceq A$ and we are done or there exists a formula-occurrence $C = D_1 \otimes D_2$ such that $X \prec C$, say $X \prec D_i$ and every empire-sequence in $\tau^X_{[A]}$ ends with a pair of steps $(\downarrow^j_\otimes)\frac{D_j}{C}, (\uparrow^i_\otimes)\frac{D_i}{C}$ with $j \neq i$, followed by a sequence of steps of the form $(\uparrow)$. Consider the subtree $\tau^C_{[A]}$ of $\tau^X_{[A]}$ ending with C. Since $\tau^X_{[A]}$ is normal, $\tau^C_{[A]}$ is normal and ends with a step of the form $(\downarrow)$. We go to "stage $p+1$, we consider $\tau^C_{[A]}$ and proceed through the "downward" part of the empire-sequence.

Each sequence of the tree $\tau^B_{[A]}$ is finite and at each stage we consider a shorter sequence. Thus the procedure must eventually end when we reach A in scanning the "upward" part of the empire-sequence. We obtain the chain

$$\mathcal{C} = A - C_n - \ldots - C_1 - B'$$

as follows. C_p is the lowermost conjunction considered during the p-th stage and $C_p - C_{p-1}$ by the axiom $\overline{\cdots, X_i, \cdots, X_j}$ considered during the p-th stage, where $i \neq j$, $X_j \prec C_{p-1}$ and $X_i \prec C_p$.

Next we need to find a *pure* chain. Let

$$\mathcal{C} = A - E_1 \otimes F_1 - \ldots - E_n \otimes F_n - B'$$

be the chain given by the above procedure and suppose $\mathcal{C}$ is not pure. Since $\tau^B_{[A]}$ is normal, then $A \neq E_i \otimes F_i \neq B'$ for all i and if $i \neq j$, then

$$D_i = D_j \text{ and } E_i = E_j, \qquad D_i = E_j \text{ and } E_i = D_j,$$

are impossible. Let $0 < i < j \leq n$. There are four possibilities:

(1) $E_i \otimes F_i \prec F_j$, or $A \prec F_j$ (2) $E_j \otimes F_j \prec E_i$, or $B' \prec E_i$
(3) $E_i \otimes F_i \prec E_j$, or $A \prec E_j$ (4) $E_j \otimes F_j \prec F_i$, or $B' \prec F_i$.

We show that cases (1) and (2) yield a cycle and in cases (3) and (4) we can consider shorter chains. In case (1) we have the cycle

$$E_j \otimes F_{j\,F_i} - E_{i+1} \otimes F_{i+1} - \ldots - E_{j-1} \otimes F_{j-1} - E_j \otimes F_j$$

or the cycle

$$E_j \otimes F_{j\,A} - E_1 \otimes F_1 - \ldots - E_{j-1} \otimes F_{j-1} - E_j \otimes F_j$$

and in case (2) the cycle

$$E_i \otimes F_i - E_{i+1} \otimes F_{i+1} - \ldots - E_{j-1} \otimes F_{j-1} -_{E_j} E_i \otimes F_i.$$

or the cycle

$$E_i \otimes F_i - E_{i+1} \otimes F_{i+1} - \ldots - E_n \otimes F_n -_{B'} E_i \otimes F_i.$$

Case (3) yields the shorter chain

$$A - \ldots - E_{i-1} \otimes F_{i-1} -_{E_i} E_j \otimes F_j - E_{j+1} \otimes F_{j+1} - \ldots B'$$

or the shorter chain

$$A - \ldots - E_{i-1} \otimes F_{i-1} -_{E_i} E_j \otimes F_j - E_{j+1} \otimes F_{j+1} - \ldots B'$$

or the shorter chain

$$A' - \ldots - E_{j+1} \otimes F_{j+1} - \ldots B'$$

where A' is $E_j \otimes F_j$, and case (4) the shorter chain

$$A - \ldots - E_i \otimes F_{i\,F_j} - E_{j+1} \otimes F_{j+1} - \ldots B'.$$

or

$$A - \ldots - B''$$

where B'' is $E_i \otimes F_i$. By repeating this argument we eventually find a pure chain as desired. ∎

5.3. Connectedness and Minimality of Paths.

The notion of a *minimal* path is considered. It is shown that if $\mathcal{S}$ is a structure and the path $\mathcal{P}$ is the set of axiom-links of $\mathcal{S}$, then the minimality of $\mathcal{P}$ is equivalent to the connectedness condition for $\mathcal{S}$.

Proposition. *Let S be a proof-structure in* **MLL**.

(i) *Let $\mathcal{P}$, $\mathcal{P}'$ be paths in S. If $\mathcal{P}'$ is a subset of $\mathcal{P}$ and $\mathcal{P}' \not\mapsto A$, then $\mathcal{P}' \not\mapsto B$, for all B belonging to $e_{\mathcal{P}}(A)$, the empire of A relative to $\mathcal{P}$.*

(ii) *Given $A, B \in \mathcal{S}$ and a path $\mathcal{P}$ in $\mathcal{S}$, if for all X such that $X \prec B$ and $\overline{\cdots X \cdots} \in \mathcal{P}$ we have $X \in e_{\mathcal{P}}(A)$, then $B \in e_{\mathcal{P}}(A)$.*

Proof. (i) By induction on the empire of A relative to $\mathcal{P}$. The induction step is immediate in the case of clauses $(i) - (iii)$ and $(v) - (vi)$. In the case of clause (iv) we use the relevance condition on conjunctions.

(ii) By induction on the logical complexity of B. ∎

Multiplicative Disjunction Lemma. *Let $\mathcal{S}(\Gamma)$ be a proof-structure in* **MLL**. *Let $\mathcal{P}$ be a path in $\mathcal{S}$. Then $\mathcal{P}$ is minimal if and only the connectedness condition relativized to $\mathcal{P}$ holds,* i.e., if for all A, B such that $\mathcal{P} \mapsto A$ and $\mathcal{P} \mapsto B$ and either $A, B \in \Gamma$ or $A \sqcup B \in \mathcal{S}$ we have $A \Diamond B$.

Proof. $(\Rightarrow)$ Suppose $\mathcal{P}$ is minimal, $A, B \in \Gamma$ or $A \sqcup B \in \mathcal{S}$, $\mathcal{P} \mapsto A$ and $\mathcal{P} \mapsto B$, but $A \notin e(B)$. Let $\mathcal{P}'$ be obtained from $\mathcal{P}$ by removing all axiom links $\overline{\cdots X_i, \cdots}$ such that $X_i \in e(B)$, for some i. We show that the set $\mathcal{P}'$ is nonempty and satisfies the relevance condition on conjunctions, i.e., $\mathcal{P}'$ is a path, contradicting the minimality of $\mathcal{P}$.

Consider any conjunction C such that $C \notin e(B)$, but C contains a door D of $e(B)$. Say $C = E \otimes F$, $D \prec F$ and let $\frac{D \quad H}{G}$ be the corresponding link. G cannot be a multiplicative conjunction, since in this case D should be B, but we assume that B is either a conclusion of $\mathcal{S}$ or a disjunct in $A \sqcup B$. Thus G is $D \sqcup H$. Then $H \notin e(B)$ and $\mathcal{P} \mapsto H$.

From part (ii) of the above Proposition it follows that we must have $X_j \notin e(B)$ for some X_j occurring in an axiom link $\overline{\cdots X_j \cdots} \in \mathcal{P}$ such that $X_j \prec H$. But then $X_k \notin e(B)$ for all k, thus $\overline{\cdots X_j \cdots} \in \mathcal{P}'$ and $\mathcal{P}' \mapsto H$. Therefore $\mathcal{P}' \mapsto F$.

Since C was an arbitrary conjunction, the relevance condition for conjuncts is satisfied and $\mathcal{P}'$ is a path for S, contradicting the minimality of $\mathcal{P}$. Therefore $A \in e(B)$. Simmetrically, we prove $B \in e(A)$.

($\Leftarrow$) Suppose there is a proper subset $\mathcal{P}'$ of $\mathcal{P}$ such that $\mathcal{P}'$ is still a path for S. Suppose $\mathcal{P} \mapsto X$ and $\mathcal{P}' \not\mapsto X$.

If $X \in \Gamma$, choose $Z \in \Gamma$ such that $\mathcal{P}' \mapsto Z$. Such Z exists, since $\mathcal{P}'$ is nonempty. Also $\mathcal{P} \mapsto Z$ since $\mathcal{P}' \subset \mathcal{P}$. Since $X, Z \in \Gamma$, we have $X \Diamond Z$, by connectedness relatively to $\mathcal{P}$. By part (i) of the previous proposition, $\mathcal{P}' \not\mapsto Z$, since $Z \in e(X)$, a contradiction.

Otherwise, we may suppose that X is premise of a link $\frac{X \quad Z}{X \circ Z}$ such that $\mathcal{P}' \mapsto Z$. Notice that $\circ$ cannot be $\otimes$, since $\mathcal{P}'$ satisfies the relevance condition on conjuctions. Thus $\circ$ must be $\sqcup$. Now $\mathcal{P} \mapsto Z$, since $\mathcal{P}' \subset \mathcal{P}$ and therefore by our assumption $X \Diamond Z$, relatively to $\mathcal{P}$. By part (i) of the above proposition, $\mathcal{P}' \not\mapsto Z$, a contradiction. ∎

Remark. In case A, B occur in Γ or if $\frac{A \quad B}{A \sqcap B}$ is a link, it is still true that $A \Diamond B$ for the relativized notion of empire implies $e_{\mathcal{P}}(B) = e_{\mathcal{P}}(A)$. Namely, $e_{\mathcal{P}}(B) \subset e_{\mathcal{P}}(A)$ follows from part (i) of the Shared Empires Lemma (section 4.3) and the remark that no step $\downarrow \frac{A}{X}$ can occur in any computation of $e_{\mathcal{P}}(B)$, since A is either a conclusion of S or a disjunct in $A \sqcup B$. Symmetrically we prove $e_{\mathcal{P}}(A) \subset e_{\mathcal{P}}(B)$.

6. Proof-Networks for Linear Logic.

We consider now an alternative representation of derivability in the fragment **MALL**. The notion of *Family of Quasi-Structures* is a development of the notion of *slicing of a proof-net* in Girard [1987b], section 6. The problem of a correct representation of logical derivability in such data-structures (see Remark 6.3. in Girard [1987b]) is solved by considering the families that result from the slicing of an arbitrary proof-structure and by providing a set of conditions that characterize *proof-network*, namely, the families representing derivation in **MALL**.

We introduce the notion of a quasi-structure, which is like a proof-structure, except that the $\sqcap$-links are unary and there are no Contraction links. Given a proof-structure $\mathcal{S}_\varphi$, we consider the family of quasi-structures that can be appropriately embedded in $\mathcal{S}_\varphi$ and that satisfy a certain maximality property. The set of all such embeddings is our interpretation of Girard's notion of a slicing. The same family of quasi-structures corresponds to two different proof-structures that differ only by a permutation of a $\otimes$-link with a $\sqcap$-link. This representation solves the problem of the uniqueness of the Normal form for **MALL**. Pairs of-structures that differ only by the permutation of a $\otimes$-link with a $\sqcap$-link correspond to a *commutative reductions* in Gentzen-Prawitz's Natural Deduction, therefore our representation is a step towards the solution of a long-standing problem of Proof-Theory.

6.1. Families of Quasi-Structures.

Definition. (i) *Structures* are defined in section (3); *embeddings* (and more generally, *maps preserving a link*) are defined in section (3.3).

(ii) A *Quasi-structure* $\mathcal{Q}$ is defined as a proof-structure for **MALL**, except that (1) there are no contraction links and (2) the $\sqcap$-links have only one premise:

$$\frac{X}{X \sqcap Y} \quad \frac{Y}{X \sqcap Y}$$

(iii) Map preserving links and embeddings are defined for Quasi-structures as for Proof-structures.

(iv) A *segment* σ in $\mathcal{S}_\varphi$ is a sequence of occurrences $C_0, \ldots, C_n$ of the same formula with the property that

(1) C_0 is not the conclusion and C_n is not the premise of a Contraction link;

(2) if $0 \leq i < n$, then C_{i+1} the conclusion of a Contraction link with premise C_i.

Given segments $\sigma_1 = C_0, \ldots, C_m$ and $\sigma_2 = D_0, \ldots, D_n$, then (with some abuse of terminology) we say that σ_1 and σ_2 are conclusions of an axiom link if and only if C_0 and D_0 are; if C_m is a premise of a link with conclusion D_0, then we say that σ_1 is a premise and σ_2 is a conclusion of the link. In this case we write

$$\frac{\sigma_1 \quad \sigma_2}{\sigma} \quad \text{or} \quad \frac{\sigma_1}{\sigma}.$$

(v) We generalize the notion of link-preservation as follows. If $\mathcal{Q}$ is a quasi-structure and $\mathcal{S}_\varphi$ is a proof-structure, consider any map $m : \mathcal{Q} \to \mathcal{S}_\varphi$. We say that m *preserves an axiom link*

$$\overline{P_1, \ldots, P_k}$$

if there are segments $\sigma_1, \ldots, \sigma_k$ in $\mathcal{S}_\varphi$ and an axiom $\overline{Q_1, \ldots, Q_h}$ such that for all $j \leq k$, $m(P_j) \in \sigma_j$ and σ_j is a conclusion of $\overline{Q_1, \ldots, Q_h}$. Moreover, we say that m *preserves a link* different from a $\sqcap$-link, of the form

$$\frac{A_1 \quad A_2}{B} \quad \text{or} \quad \frac{A}{B},$$

if there are segments σ_1, σ_2, σ and a link in $\mathcal{S}_\varphi$ of the same kind of the link in question, such that

$$\frac{\sigma_1 \quad \sigma_2}{\sigma} \quad \text{or} \quad \frac{\sigma_1}{\sigma}.$$

Finally, we say that m *preserves a $\sqcap$-link*

$$\frac{A}{B}$$

in $\mathcal{Q}$, if there is a $\sqcap$-link in $\mathcal{S}_\varphi$ if there are segments σ_1, σ_2, σ and a $\sqcap$-link in $\mathcal{S}_\varphi$ such that

$$\frac{\sigma_1 \quad \sigma_2}{\sigma}$$

$B \in \sigma$ and for $i = 1$ or 2, $A \in \sigma_i$. An injection m preserving links is called an *embedding* if $m(A) \sim A$.

(vi) Let **Struct** be the category whose objects $\mathcal{O}$ are either proof-structures or quasi-structures and whose morphisms are embeddings.

(vii) Let **Struct*** be the category whose objects $(\mathcal{O}, A)$ are structures with a selected conclusion A and whose morphisms are embeddings preserving links and selected formula-occurrences — we say that $m : (\mathcal{O}, A) \to (\mathcal{O}', A')$ preserves the selected formula-occurrence if $m(A) = A'$.

Remark. Let $\iota : \mathcal{O} \to \mathcal{O}'$ be a morphism of **Struct**. ι is an embedding of quasi-structures, if $\mathcal{O}'$ is a quasi-structure, an embedding of proof-structures if both $\mathcal{O}$ and $\mathcal{O}'$ are proof-structures. In such cases links are preserved in the sense of section (3.3).

We consider the assignments

Times $$(\mathcal{O}_1, A), (\mathcal{O}_2, B) \mapsto (\mathcal{O}_1 \otimes \mathcal{O}_2, A \otimes B);$$

Plus $$(\mathcal{O}, A) \mapsto (\frac{\mathcal{O}}{A \oplus B}, A \oplus B),$$

Some $$(\mathcal{O}, A) \mapsto (\frac{\mathcal{O}}{\exists x.A}, \exists x.A).$$

Fact. *The assignment* Times *determines a functor* **Struct**$^* \times$ **Struct**$^* \to$ **Struct***. *The assignments* Plus *and* Some *determine a functor* **Struct**$^* \to$ **Struct***. ∎

Given a proof-structure, we consider all quasi-structures that are embedded in the given proof-structure and we select the family of those which are maximal in the following sense.

Definition. (i) Given $\iota : \mathcal{Q} \to \mathcal{S}_\varphi$ in the category **Struct** and $A \in \mathcal{Q}$, let σ_A be the segment σ in $\mathcal{S}_\varphi$ such that $\iota(A) \in \sigma$. Let $\mathcal{P}(\mathcal{S}_\varphi)$ be the set of subsets of $\mathcal{S}_\varphi$. We define a map $\iota_\sigma : \mathcal{Q} \to \mathcal{P}(\mathcal{S}_\varphi)$ by the assignment $A \mapsto \sigma_A$.

(ii) Let $D_1,\ldots,D_n$ be the conclusions of $\mathcal{Q}$ and let $C_1,\ldots,C_m$ be the conclusions of $\mathcal{S}_\varphi$. We say that a morphism $\iota : \mathcal{Q} \to \mathcal{S}_\varphi$ in the category **Struct** *strongly preserves the conclusions* if $m = n$ and for all $i \le n$, $C_i \in \iota_\sigma(D_i)$.

(iii) We say that a morphism $\iota : \mathcal{Q} \to \mathcal{S}_\varphi$ in the category **Struct** and its domain $\mathcal{Q}$ are *maximal* if the following condition is satisfied:

for every $\iota' : \mathcal{Q}' \to \mathcal{S}_\varphi$, if ι' strongly preses conclusions, then $\mathcal{Q} \subset \mathcal{Q}'$ implies $\mathcal{Q} = \mathcal{Q}'$.

(iv) We let *Fam* $\mathcal{S}_\varphi$ be

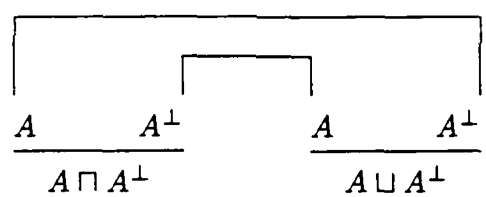
$$\{\mathcal{Q} : \text{ the set-inclusion map is a maximal morphism } \iota : \mathcal{Q} \to \mathcal{S}_\varphi \text{ in } \mathbf{Struct}\}.$$

and we let *Fam* be

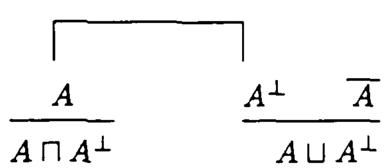
$$\{\mathcal{Q} \xrightarrow{\iota} \mathcal{S}_\varphi \;:\; \iota \text{ is the set-inclusion map and is a maximal morphism in } \mathbf{Struct}\}.$$

(iv) Given a proof-structure $\mathcal{S}_\varphi$, we say that *Fam strongly preserves the axioms* (in $\mathcal{S}_\varphi$) if for each $\mathcal{Q}$ in *Fam* $\mathcal{S}_\varphi$ there is an $\iota : \mathcal{Q} \to \mathcal{S}_\varphi$ strongly preserving axioms — i.e., for every axiom $\overline{P_1, \ldots, P_k}$ of $\mathcal{Q}$ there is an axiom $\overline{P_1', \ldots, P_k'}$ of $\mathcal{S}_\varphi$ such that $P_i' \in \iota_\sigma(P_i)$, for all $i \leq k$.

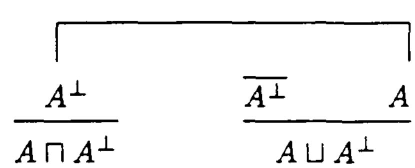
Example. Consider the following proof-structure $\mathcal{S}_\varphi$.



Fam $\mathcal{S}_\varphi$ consists of the following quasi-structures $\mathcal{Q}_1$ and $\mathcal{Q}_2$:



and



Notice that here $\mathcal{Q}_1$ and $\mathcal{Q}_2$ must contain non-logical axioms and *Fam* cannot strongly preserve the axioms in $\mathcal{S}_\varphi$.

6.2. Empires in a Family of Structures.

The notion of empire is generalized to families of quasi-structures. Conditions are given to characterize the class of families that represent correct derivations. Such families will be called proof-networks. It seems that this representation fulfills Girard's description as the 'absolute limit for a parallelization of the syntax' (Girard [1987b] p.94), since the vicious circle and connectedness conditions for a family are reduced to independent verifications of the same conditions for each quasi-structure of the family.

Definitions. (i) Given $A \in \mathcal{S}_\varphi$, consider the set of all the formula occurrences A', A'',... in all the $\mathcal{Q} \in$ *Fam* $\mathcal{S}_\varphi$. The relation that holds between A' and A'' if and only if there are ι', ι'' in *Fam* such that $\iota'(A') = \iota''(A'') = A$ is an equivalence relation. Let $[A]$ be the corresponding equivalence class of formula-occurrences. For all $\mathcal{Q} \in$ *Fam* $\mathcal{S}_\varphi$, let $A_\mathcal{Q}$ be the element in $\mathcal{Q} \cap [A]$, if such an element exists. Thus when we write $A_\mathcal{Q}$ we assume that $\mathcal{Q} \cap [A] \neq \emptyset$.

(*ii*) For any $\mathcal{F}$ and any $\mathcal{Q} \in \mathcal{F}$ the definition of empire of A in $\mathcal{Q}$ is as in section (3.4), except that clause (*iv*) applies now to the $\sqcap$-links links and there are no contraction links. We say that $\mathcal{F}$ satisfies the vicious-circle [the connectedness conditions] if every $\mathcal{Q} \in \mathcal{F}$ satisfies it.

(*iii*) Let $\mathcal{F} =$ *Fam* $\mathcal{S}_\varphi$. The *empire* $e[A]$ of an *equivalence class* $[A]$ is a set of equivalence classes, defined as follows: $[B] \in e[A]$ in $\mathcal{F}$ if and only if for every $\mathcal{Q} \in \mathcal{F}$ we have $B_\mathcal{Q} \in e(A_\mathcal{Q})$, i.e, if and only if for every $\mathcal{Q} \in \mathcal{F}$ such that $B_\mathcal{Q}$ exists, also $A_\mathcal{Q}$ exists and $B_\mathcal{Q} \in e(A_\mathcal{Q})$ in $\mathcal{Q}$.

(*iv*) The *box condition* for a family $\mathcal{F} =$ *Fam* $\mathcal{S}_\varphi$ of quasi-structures is the condition: *Fam strongly preserves the axioms and for every* $D, D' \in \mathcal{S}_\varphi$ *if* $\varphi(D) = \varphi(D')$ *then* $[D]\Diamond[D']$. Notice that by definition of φ, the formula-occurrences D and D' considered here are doors of $e(X_i)$ in $\mathcal{S}_\varphi$, for some link $\frac{X_1 \quad X_2}{X_1 \sqcap X_2}$.

(*v*) Let $\mathcal{F} =$ *Fam* $\mathcal{S}_\varphi$ and let a and b be eigenvariables associated with $\bigwedge x.A$ and $\bigwedge y.A'$, respectively, where $\bigwedge x.A$ occurs in $\mathcal{Q}_1$ and $\bigwedge y.A'$ occurs in $\mathcal{Q}_2$ and $\mathcal{Q}_1, \mathcal{Q}_2 \in \mathcal{F}$. We say that a and b are *identified* if and only if $[A(a)] = [A'(b)]$ in $\mathcal{F}$. When we speak of an eigenvariable a in $\mathcal{F}$, it must be understood that we refer to the equivalence class of eigenvariables identified with a.

(vi) Let $\mathcal{F} = Fam\ \mathcal{S}_\varphi$. We define the relations $<^0$ and $<_t$ between (equivalence classes of) eigenvariables in $\mathcal{F}$ as follows. Given $[\bigwedge x.A(x)]$ and $[\bigwedge y.B(y)]$ in $\mathcal{F}$, let a and b be the (equivalence classes of) eigenvariables associated with $\bigwedge x.A$ and $\bigwedge y.B$.

Then $a <^0 b$ if and only if *the eigenvariable b occurs outside and inside $\epsilon[A(a)]$*.

Let $<_t$ be the transitive closure of $<^0$.

(vii) The *parameters* condition is the requirement that $<_t$ in $\mathcal{F}$ *is strict.*

($viii$) Let $\mathcal{F} = Fam\ \mathcal{S}_\varphi$. $\mathcal{F}$ is a *proof network* if every quasi-structure $\mathcal{Q}$ in $\mathcal{F}$ satisfies the vicious circle and connectedness conditions and if $\mathcal{F}$ satisfies the box and parameter conditions.

6.3. Equivalence Theorem.

The proof of the following facts is a straightforward induction on the definition of $\mathcal{S}_\varphi$.

Facts. *Let $\mathcal{S}_\varphi$ be inductively generated* (see clauses (0) - (7) in sections 3.1 and 3.2).

(i) (clauses (0), (1), (1)' *and* (1)") *If $\mathcal{S}_\varphi$ is an axiom, then $Fam\ \mathcal{S}_\varphi = \{\mathcal{S}_\varphi\}$.*

(ii) (clause (2)) *If $\mathcal{S}_\varphi$ has the form*

$$\mathcal{S}'_{\varphi'}\ {}_{A\otimes B}\ \mathcal{S}''_{\varphi''},$$

where $\mathcal{S}'_{\varphi'}$ and $\mathcal{S}''_{\varphi''}$ do not share any formula-occurrence, then $Fam\ \mathcal{S}_\varphi$ is

$$\{\mathcal{Q}'\ {}_{A\otimes B}\ \mathcal{Q}'' : \mathcal{Q}' \in Fam\ \mathcal{S}'_{\varphi'} \text{ and } \mathcal{Q}'' \in Fam\ \mathcal{S}''_{\varphi''}\}.$$

and for each $\mathcal{Q}'\ {}_{A\otimes B}\ \mathcal{Q}''$, $\mathcal{Q}'$ and $\mathcal{Q}''$ do not share any formula-occurrence.

(iii) (clauses (5) *and* (6)) *If $\mathcal{S}_\varphi$ has the form*

$$\frac{\mathcal{S}'_{\varphi'}}{\exists x.A(x)} \quad \text{or} \quad \frac{\mathcal{S}'_{\varphi'}}{A \oplus B},$$

then $Fam\ \mathcal{S}_\varphi$ i

$$\left\{\frac{\mathcal{Q}'}{\exists x.A(x)} : \mathcal{Q}' \in Fam\ \mathcal{S}'_{\varphi'}\right\} \quad \text{or} \quad \left\{\frac{\mathcal{Q}'}{A \oplus B} : \mathcal{Q}' \in Fam\ \mathcal{S}'_{\varphi'}\right\}$$

respectively.

(iv) (clauses (3) *and* (4)) *If* $\mathcal{S}_\varphi$ *has the form*

$$\frac{\mathcal{S}'_{\varphi'}}{A \sqcup B}, \quad \text{or} \quad \frac{\mathcal{S}'_{\varphi'}}{\forall x.A(x)},$$

where the eigenvariable associated with $\forall x.A(x)$ *does not occur in the conclusions of* $\mathcal{S}_\varphi$, *then Fam* $\mathcal{S}_\varphi$ *is*

$$\left\{\frac{\mathcal{Q}'}{A \sqcup B} : \mathcal{Q}' \in \mathit{Fam}\ \mathcal{S}'_{\varphi'}\right\} \quad \text{or} \quad \left\{\frac{\mathcal{Q}'}{\forall x.A(x)} : \mathcal{Q}' \in \mathit{Fam}\ \mathcal{S}'_{\varphi'}\right\}.$$

(v) (clause (7)) *If* $\mathcal{S}'_{\varphi'}(\Lambda', A)$ *and* $\mathcal{S}''_{\varphi''}(\Lambda'', B)$ *are inductive proof-structures that do not share any formula-occurrence, and* Λ, Λ' *and* Λ'' *are different occurrences of the same sequence* $D_1, D_2, \ldots, D_n$ *and* $\mathcal{S}_\varphi$ *is* $\mathcal{S}' \cup \mathcal{S}'' \cup \Lambda \cup \{A \sqcap B\}$ *with the addition of the links*

$$\frac{D'_1 \quad D''_1}{D_1} \quad \cdots \quad \frac{D'_n \quad D''_n}{D_n} \qquad \frac{A \quad B}{A \sqcup B}$$

then Fam $\mathcal{S}_\varphi$ *is*

$$\left\{\frac{\mathcal{Q}'}{A \sqcap B} : \mathcal{Q}' \in \mathit{Fam}\ \mathcal{S}'_{\varphi'}\right\} \cup \left\{\frac{\mathcal{Q}'}{A \sqcap B} : \mathcal{Q}' \in \mathit{Fam}\ \mathcal{S}''_{\varphi''}\right\}.$$

(vi) *Fam* $\mathcal{S}_\varphi$ *preserves the axioms of* $\mathcal{S}_\varphi$. ∎

Proposition. *Let* $\mathcal{S}_\varphi$ *be a proof-structure for* **MALL** *and suppose* Fam $\mathcal{S}_\varphi$ *is a proof-network. Then for each* $\iota : \mathcal{Q} \to \mathcal{S}_\varphi$ *in* Fam

(1) *for each link of* $\mathcal{S}_\varphi$ *different from a* $\sqcap$ *and a Contraction link, the premises of the link are in* $\iota_\sigma(\mathcal{Q})$ *if and only if the conclusions are;*

(2) *for all* $\sqcap$- *and Contraction links of* $\mathcal{S}_\varphi$

(2.1) *one and only one premise of such a link is in* $\iota_\sigma(\mathcal{Q})$ *if and only if the conclusions are;*

(2.2) *if* $\varphi(Y) = \varphi(Z)$, *then* Y *is in* $\iota_\sigma(\mathcal{Q})$ *if and only if* Z *is;*

(3) $\mathcal{S}_\varphi$, as a set, *is the union of all the* $\mathcal{Q}$ *in* Fam $\mathcal{S}_\varphi$.

Proof. The proof is straightforward from the definition of *Fam*, by the property of maximality of each embedding in it. The Box condition for proof-network is used to obtain (2.2). ∎

We can prove now the

Equivalence Theorem. (i) *Let $\mathcal{S}_\varphi$ be a proof-structure satisfying the vicious-circle, connectedness and box conditions. Then for all $X, Y \in \mathcal{S}_\varphi$, $Y \in e(X)$ if and only if $[Y] \in e[X]$ in $Fam\mathcal{S}_\varphi$.*

(ii) *$\mathcal{S}_\varphi$ is a proof-net if and only if $Fam\ \mathcal{S}_\varphi$ is a proof-network.*

Proof. If $\mathcal{S}_\varphi$ satisfies the vicious-circle, connectedness and box conditions, then it is inductively generated as a propositional structure. Hence we can use induction on $\mathcal{S}_\varphi$ to prove part (i) and to show that if the vicious circle, connectedness and box conditions hold for $\mathcal{S}_\varphi$, then they hold also for $Fam\ \mathcal{S}_\varphi$. To finish the proof of part (ii), "only if" direction, we use part (i) and conclude that the parameters condition is preserved in the passage from $\mathcal{S}_\varphi$ to $Fam\ \mathcal{S}_\varphi$. To prove (ii), "if" direction, we show that if $\mathcal{F} = Fam\ \mathcal{S}_\varphi$ is a proof-network, then $\mathcal{S}_\varphi$ satisfies the box condition. By induction on the number of $\sqcap$-links in $\mathcal{S}_\varphi$, we reduce to the case of substructures with nonlogical axioms in the multiplicative fragment with $\oplus$. In this fragment proof-structures and quasi-structures coincide and $Fam\ \mathcal{S} = \{\mathcal{S}\}$. Hence $\mathcal{S}_\varphi$ satisfies the vicious circle and connectedness condition. The fact that $\mathcal{S}_\varphi$ satisfies the parameters condition is then immediate by (i).

Part (i). By induction on the definition of $\mathcal{S}_\varphi$ we show $Y \in e(X)$ if and only if $[Y] \in e[X]$. If $\mathcal{S}$ is an axiom, this is clear. If $\mathcal{S}$ results from a clause (3) – (6) in the inductive definition of proof-structure, then this fact is immediate from the induction hypothesis.

Clause (2). If $\mathcal{S}_\varphi$ is $\mathcal{S}^1_{\varphi_1}\,{}_{A}{\otimes}_{B}\,\mathcal{S}^2_{\varphi_2}$, let $\mathcal{F}^1 = Fam\ \mathcal{S}^1_{\varphi_1}$ and $\mathcal{F}^2 = Fam\ \mathcal{S}^2_{\varphi_2}$. By Fact ($ii$) above, every quasi-structure in $\mathcal{F} = Fam\ \mathcal{S}_\varphi$ has the form $\mathcal{Q}^1\,{}_{A}{\otimes}_{B}\,\mathcal{Q}^2$, where $\mathcal{Q}^1 \in \mathcal{F}^1$ and $\mathcal{Q}^2 \in \mathcal{F}^2$.

Suppose X and Y occur in $\mathcal{S}^1_{\varphi_1}$. By induction hypothesis, $Y \in e(X)$ in $\mathcal{S}_\varphi$ if and only if $[Y] \in e[X]$ in $\mathcal{F}^1$, i.e., if and only if for all $\mathcal{Q} \in \mathcal{F}^1$, if $[Y] \cap \mathcal{Q} \neq \emptyset$, then $Y_{\mathcal{Q}} \in e(X_{\mathcal{Q}})$. Since X and Y both occur in $\mathcal{Q}^1$, by the Fact (ii) in section (3.6) we have $Y \in e(X)$ in $\mathcal{Q}^1\,{}_{A}{\otimes}_{B}\,\mathcal{Q}^2$ if and only if $Y \in e(X)$ in $\mathcal{Q}^1$. Thus in this case the result follows. The same argument holds if X and Y occur in $\mathcal{S}^2$.

Now suppose that Y occurs in $\mathcal{S}^2$, X occurs in $\mathcal{S}^1$ and $Y \in e(X)$ in $\mathcal{S}$. Then by the Fact (i) in section (3.6) $X \neq A$ and $A \otimes B \in e(X)$ and by induction hypothesis

$A_{\mathcal{Q}} \in e(X_{\mathcal{Q}})$ for all $\mathcal{Q} \in \mathcal{F}^1$. It follows that $(A \otimes B)_{\mathcal{Q}} \in e(X_{\mathcal{Q}})$ for all $\mathcal{Q} \in \mathcal{F}$. Since $e(B) = \mathcal{Q}^2$, by connectedness for $\mathcal{Q}^2$, we have $\mathcal{Q}^2 \subset e(X)$ by induction on $e(B)$ in $\mathcal{Q} = \mathcal{Q}^1 {}_A\!\otimes_B \mathcal{Q}^2$. Since this holds for all $\mathcal{Q} \in \mathcal{F}$, we have $[Y] \in e[X]$ in $\mathcal{F}$, as desired.

Conversely, if $Y \notin e(X)$ in $\mathcal{S}$, then it is easy to show that also $A \notin e(X)$, thus by induction hypothesis there is a $\mathcal{Q}^1$ in $\mathcal{F}^1$ such that either $[X] \cap \mathcal{Q}^1 = \emptyset$ or $A_{\mathcal{Q}^1} \notin X_{\mathcal{Q}^1}$. Choose $\mathcal{Q}^2$ such that $Y_{\mathcal{Q}^2}$ exists. Then by the Fact (ii) above, $\mathcal{Q} = \mathcal{Q}^1 {}_A\!\otimes_B \mathcal{Q}^2$ is in $\mathcal{F}$. Also $\mathcal{Q}$ is such that $Y_{\mathcal{Q}}$ exists, but if $X_{\mathcal{Q}}$ exists, then $Y_{\mathcal{Q}} \notin X_{\mathcal{Q}}$, by the Fact (i) in section (3.6). We conclude that $[Y] \notin e[X]$.

Clause (7). If $\mathcal{S}_\varphi(C_1, \ldots, C_n, A \sqcap B)$ results from $\mathcal{S}^1$ and $\mathcal{S}^2$ by clause (7), then the lowermost links of $\mathcal{S}$ are

$$\frac{C_1' \quad C_1''}{C_1} \quad \cdots \quad \frac{C_n' \quad C_n''}{C_n} \quad \frac{A \quad B}{A \sqcap B}$$

and $\mathcal{F} = \mathit{Fam}\ (\mathcal{S}_\varphi)$ consists of quasi-structures of the forms

$$\overset{\mathcal{Q}^1}{C_1 \ldots C_n} \quad \frac{A}{A \sqcap B} \qquad \text{and} \qquad \overset{\mathcal{Q}^2}{C_1 \ldots C_n} \quad \frac{B}{A \sqcap B}$$

where $\mathcal{Q}^1 \in \mathcal{F}^1 = \mathit{Fam}\ (\mathcal{S}^1)$ and $\mathcal{Q}^2 \in \mathcal{F}^2 = \mathit{Fam}\ (\mathcal{S}^2)$. By the Substructure Theorem for **MALL** (section 4.5), for all $X, Y \in \mathcal{S}$ we have the following cases:

(1) if both X and Y occur in $\mathcal{S}^i$ then $Y \in e(X)$ in $\mathcal{S}$ if and only if $Y \in e(X)$ in $\mathcal{S}^i$, for $i = 1, 2$;

(2) if X occurs in $\mathcal{S}^1$ and Y occurs in $\mathcal{S}^2$, then $Y \notin e(X)$ and $X \notin e(Y)$;

(3) if Y occurs in $\mathcal{S}^1$ or in $\mathcal{S}^2$ and X is one of C_1,...,C_n, $A \sqcap B$, then $Y \in e(X)$ and $X \notin e(Y)$.

It is easy to check that the corresponding conditions hold for $\mathcal{F}$. For instance, if $[Y] \in \mathcal{F}^1$ and $[X] = [C_i]$, then $\mathcal{Q}^1 \subset e(X_{\mathcal{Q}^1})$ for all $\mathcal{Q}^1 \in \mathcal{F}^1$, by connectedness and $[X] \notin e[Y]$, since $[X] \cap \mathcal{Q}^2 \neq \emptyset$ and $[Y] \cap \mathcal{Q}^2 = \emptyset$ for all $\mathcal{Q}^2 \in \mathcal{F}^2$.

Part (ii), (*only if*). Let $\mathcal{S}_\varphi(\Gamma)$ be a proof-net for **MALL** and let $\mathcal{F}(\Gamma) = \mathit{Fam}\ (\mathcal{S}_\varphi)$. The inductive definition of $\mathcal{S}_\varphi$ induces an inductive generation of every quasi-structure $\mathcal{Q} \in \mathcal{F}$, by the Facts $(i) - (v)$ above. It follows that every $\mathcal{Q} \in \mathit{Fam}\ (\mathcal{S}_\varphi)$

is an inductive structure in the fragment **MLL** extended with one-premise rules, which are irrelevant for the verification of the vicious circle and connectedness conditions. Therefore the structure theorem in section (3.6) still applies and each $\mathcal{Q}$ satisfies the vicious circle and connectedness condition. The box condition follows the Fact (v) above, from the Box-Door Lemma (section 4.5) applied to $\mathcal{S}_\varphi$ and from part (i). The parameters condition is immediate from part (i).

Part (ii), (*if*). Given a proof-network $\mathcal{F} = \mathit{Fam}\,\mathcal{S}_\varphi$, the conclusions Γ of $\mathcal{S}_\varphi$ belong to $\iota_\sigma(\mathcal{Q})$, for every $\iota : \mathcal{Q} \to \mathcal{S}_\varphi$ in *Fam*, by definition. Thus for any $C \in \Gamma$, $[C] \cap \mathcal{Q} \neq \emptyset$. Consider any link

$$\frac{X' \quad X''}{X}$$

in $\mathcal{S}_\varphi$ with the following property:

$(*)$ $\quad [X] \cap \mathcal{Q} \neq \emptyset$ *for all* $\mathcal{Q} \in \mathcal{F}$, *but* $[X'] \cap \mathcal{Q} = \emptyset$ *for some* $\mathcal{Q}$.

Then the link in question must be a $\sqcap$- or a Contraction link, by parts (1) and (2.1) of the above Proposition. By part (2.2) of the same Proposition, X belongs to a sequence Δ of the form $C_1, \ldots, C_n, A \sqcap B$, where $C_1, \ldots, C_n$ are conclusions of a Contraction link and $A \sqcap B$ is the conclusion of a $\sqcap$-link. It follows immediately from (2.2) that every formula-occurrence in Δ is a conclusion of a link with the property $(*)$. Let

$$\Delta_1, \ldots, \Delta_p$$

be all the distinct sequences $C_{i,1}, \ldots, C_{i,n_i}, A_i \sqcap B_i$ whose elements are conclusions of links with the property $(*)$. It follows immediately from (2.2) that if $X \in \Delta_i$ and $Y \in \Delta_j$, with $i \neq j$, then we cannot have $X \prec Y$ nor $Y \prec X$.

Choose one of the Δ_i, say $\Delta = C_1, \ldots, C_n, A \sqcap B$, and let $\mathcal{S}'_{\varphi'}$ be the structure whose formula-occurrences are those in Δ together with their ancestors, and whose links and flagging function are the restrictions of the links and of the flagging functions of $\mathcal{S}_\varphi$ to the set in question. We need to show that

(I) $\mathcal{S}'_{\varphi'}$ *is a substructure of* $\mathcal{S}_\varphi$.

To prove (I) it is enough to show that the set-inclusion map $\iota' : \mathcal{S}'_{\varphi'} \to \mathcal{S}_\varphi$ strongly preserves the axioms. Let $\overline{P_1, \ldots, P_n}$ be an axiom of $\mathcal{S}_\varphi$ that is not strongly

preserved by ι', say $P_1 \prec D \in \Delta$ and $P_n \not\prec \Delta$. Every $D \in \Delta$ is the conclusion of a link

$$\frac{D' \quad D''}{D},$$

thus either $P_1 \prec D'$ or $P_1 \prec D''$. By the definition of *Fam* $(\mathcal{S}_\varphi)$ and by part (3) of the above Proposition, there is a $\iota : \mathcal{Q} \to \mathcal{S}_\varphi$ such that $P_1 \notin \iota_\sigma(\mathcal{Q})$ and $P_n \in \iota_\sigma(\mathcal{Q})$, hence ι does not strongly preserve $\overline{P_1, \ldots, P_n}$. This contradicts the box condition for *Fam* $(\mathcal{S}_\varphi)$, The proof of (I) is finished.

Let $\Lambda_1 = C'_1, \ldots, C'_n, A$ and $\Lambda_2 = C''_1, \ldots, C''_n, B$, be the sets of formula occurrences in $\mathcal{S}_\varphi$ flagged with A, and with B, respectively. Let $\mathcal{S}^1_{\varphi_1}$ and $\mathcal{S}^2_{\varphi_2}$ be the structures defined as follows. The formula-occurrences of $\mathcal{S}^i_{\varphi_i}$ are those in Λ_i together with their ancestors, and the links and flagging function are the restrictions of the links and of the flagging functions of $\mathcal{S}_\varphi$ to the set in question. By an argument similar to that of the previous paragraph we can show that the axioms are strongly preserved by the inclusion of $\mathcal{S}^i_{\varphi_i}$ in $\mathcal{S}'_{\varphi'}$, i.e., that $\mathcal{S}^i_{\varphi_i}$ is a substructure of $\mathcal{S}'_{\varphi'}$ and of $\mathcal{S}_\varphi$, for $i = 1, 2$. We need to show that

(II) *For $i = 1, 2$, $\mathcal{F}_i = Fam\ \ \mathcal{S}^i_{\varphi_i}$ is a proof-network.*

Notice that every quasi-structure $\mathcal{Q}_i$ in $\mathcal{F}_i = Fam\ \mathcal{S}^i_{\varphi_i}$ is a substructure of a quasi-structure $\mathcal{Q}$ in $\mathcal{F} = Fam\ \mathcal{S}_\varphi$. It is immediate that every quasi-structure in $\mathcal{F}_1$, $\mathcal{F}_2$ satisfies the vicious circle condition, since every structure in $\mathcal{F}$ does.

We need to show that the connectedness condition is satisfied in each $\mathcal{Q}' \in \mathcal{F}_i$, $i = 1, 2$. Suppose C and D are either premises of a $\sqcup$-link or formulas flagged with A in $\mathcal{Q}' \in \mathcal{F}_1$. $\mathcal{Q}'$ is a substructure of a quasi-structure $\mathcal{Q} \in \mathcal{F}$ and $C \Diamond D$ in $\mathcal{Q}$ by the Box condition. By the proof of the Lemma in section (5.2), there is a pure chain $C'_C - \mathcal{C} -_D D'$ where each $X \in \mathcal{C}$ has the form $C_i \otimes D_i$. We need to show that every such $C_i \otimes D_i$ occurs in $\mathcal{Q}'$. Suppose not. Then we have $E \prec C_i$ and $F \prec D_i$ such that E and F are flagged with A. As a consequence, $E \Diamond F$, and again by the same Lemma we obtain a chain $E'_E - \mathcal{C}' -_F F'$, thus a cycle $(C_i \otimes F_i)_E - \mathcal{C}' -_F (C_i \otimes F_i)$ in $\mathcal{Q}$, a contradiction. Thus $\mathcal{Q}'$ satisfies the connectedness condition, as required.

It is immediate that *Fam* strongly preserves the axioms in $\mathcal{F}_1, \mathcal{F}_2$. An argument similar to that in the previous paragraph can be used to conclude the proof that *Fam* $\mathcal{S}^i_{\varphi_i}$ satisfies the box condition. The proof of (II) is concluded.

Let $\overline{\mathcal{S}}_{\overline{\varphi}}$ be the substructure of $\mathcal{S}_{\varphi}$ complementary to $\mathcal{S}'_{\varphi'}$. We prove

(III) Fam $\overline{\mathcal{S}}_{\overline{\varphi}}$ *is a proof-network.*

It is easy to show that $\overline{\mathcal{F}}$ satisfies the connectedness condition: simply notice that every $\overline{\mathcal{Q}}$ in $\overline{\mathcal{F}}$ is the complementary substructure of a quasi-structure $\mathcal{Q}' \in \mathcal{F}'$ in a quasi-structure $\mathcal{Q} \in \mathcal{F}$, and apply the Substructure Theorem for **MLL** (extended with unary links) (section 4.3).

To prove that every $\overline{\mathcal{Q}} \in \overline{\mathcal{F}}$ satisfies the vicious circle condition, we assume that there is a cycle in $\overline{\mathcal{Q}}$ and we may assume that in such a cycle

$$\mathcal{C}: \qquad {}_U Z_V - \ldots - X_{C'} -_{C''} Y - \ldots -_U Z_V,$$

C', C'' belong to the nonlogical axiom $\overline{\Delta}$. Furthermore, we may assume that only X and Y are connected by the nonlogical axiom $\overline{\Delta}$ (otherwise, we may shorten the cycle). Now C', C'' are flagged with A, thus $C' \diamond C''$ in $\mathcal{Q}$ by the box condition. By the proof of the Lemma in section (5.2), there is a pure chain

$$X'_{C'} - \mathcal{C}' -_{C''} Y'$$

and we may suppose that such a chain occurs inside the substructure $\mathcal{Q}'$ of $\mathcal{Q}$. By the choice of $\mathcal{C}$ no element of $\mathcal{C}'$ other than X' and Y' is an ancestor or a successor of any element of $\mathcal{C}$ other than X and Y. Therefore we may construct a cycle

$$\mathcal{C}'': \qquad {}_U Z_V - \ldots - X_{C'} - \mathcal{C}' -_{C''} Y - \ldots -_U Z_V,$$

in $\mathcal{Q}$, a contradiction.

To prove that $\overline{\mathcal{F}}$ satisfies the box condition, it is easy to show that $[D]\diamond[D']$ in *Fam* $(\overline{\mathcal{S}}_{\overline{\varphi}})$, for all D, D' in $(\overline{\mathcal{S}}_{\overline{\varphi}})$ such that $\overline{\varphi}(D) = \overline{\varphi}(D')$: an argument similar to that in the proof of the connectedneas condition for $\overline{\mathcal{F}}$ applies here as well. Thus it is enough to show that *Fam* strongly preserves the axioms in $(\overline{\mathcal{S}}_{\overline{\varphi}})$. Now the axioms of $(\overline{\mathcal{S}}_{\overline{\varphi}})$ are either axioms of $\mathcal{S}_{\varphi}$ or $\overline{\Delta}$.

Let $\overline{P_1, \ldots, P_n}$ be an axiom of $\mathcal{S}_{\varphi}$ such that $P_1 \in \overline{\mathcal{Q}}$ and $P_n \notin \overline{\mathcal{Q}}$. The only possibility is that $P_n \prec \Delta$, say $P_n \prec \Lambda_1$. By an argument similar to that of part (I) we find a $\mathcal{Q} \in \mathcal{F}$ such that $P_1 \in \mathcal{Q}$ and $P_n \notin \mathcal{Q}$, contradicting the Box condition for $\mathcal{F}$.

Suppose *Fam* does not strongly preserve the axiom $\overline{\Delta}$, i.e., for some Ξ_1, Ξ_2 and for some $\overline{Q}$ in $\overline{\mathcal{F}}$, $\Delta = \Xi_1 \cup \Xi_2$, $\Xi_1 \subset \overline{Q}$ and $\Xi_2 \cap \overline{Q} = \emptyset$. This implies that in $\overline{\mathcal{S}_{\overline{\varphi}}}$, for every formula occurrence $X \in \Xi_2$ there is some sequence $\Delta' = Y_1,\ldots,Y_\ell,Z_1 \sqcap Z_2$, different from Δ, such that $X \prec \Delta'$. The same must be true in $\mathcal{S}_\varphi$, and this contradicts the choice of Δ. The proof of (III) is finished.

We can therefore apply the induction hypothesis to $\mathcal{F}_1$, $\mathcal{F}_2$ and $\overline{\mathcal{F}} = Fam\ (\overline{\mathcal{S}}_{\overline{\varphi}})$, since by (II) and (III) they are proof-networks with lower number of $\sqcap$-links. We conclude that $\mathcal{S}^1_{\varphi_1}$, $\mathcal{S}^2_{\varphi_2}$ and $(\overline{\mathcal{S}}_{\overline{\varphi}})$ satisfy the propositional conditions for proof-nets. Since $\mathcal{S}'_{\varphi'}$ results from $\mathcal{S}^1_{\varphi_1}$ and $\mathcal{S}^2_{\varphi_2}$ by an application of clause (7) in the inductive definition of proof-structures (section 3.2), we conclude that $\mathcal{S}'_{\varphi'}$ satisfies the propositional conditions for proof-net too. Since $\mathcal{S}_\varphi$ results by substituting the proof-structure $\mathcal{S}'_{\varphi'}$ for the nonlogical axiom Δ_i of the proof-structure $(\overline{\mathcal{S}}_{\overline{\varphi}})$, it follows that $\mathcal{S}_\varphi$ satisfies the propositional conditions for proof-nets, by the Substructure Theorem for **MALL**, section (4.5). ■

6.4. Notes on Noncommutative Linear Logic.

We sketch a treatment of the systems **NCMALL** for noncommutative linear logic. **Definition.** A proof-structure $\mathcal{S}$ for (*noncommutative*) first order **NCMLL** is defined as a proof-structure for commutative **MLL**, together with a cyclic order $<$ of the conclusions of $\mathcal{S}$ satisfying conditions (1), (2) in section (3) and in addition the following condition (3)'. Extend $<$ to a lexicographical ordering on $\mathcal{S}$, by letting $X < Y$ if and only if $U < V$, where U and V are conclusions and $X \preceq U$ and $Y \preceq V$.

(3)' *planarity condition*: (see Girard [1989a], II.9) *If*

$$\overline{\ldots X_i \ldots X_j \ldots} \quad \text{and} \quad \overline{\ldots Y_m \ldots Y_n \ldots}$$

are any pair of distinct axiom links then in the induced ordering we cannot have

$$X_i < Y_m < X_j < Y_n.$$

The planarity condition for $\mathbf{NCMLL}^-$ can be summarized by saying that in the plane of the page in which a proof-structure is written, there is no crossing of the axiom links. Notice that this property is preserved, if a *cyclic* permutation is applied

to the conclusions of $\mathcal{S}$. In the case **NCMLL**, the ordering of the occurrences of $\perp$ is irrelevant. More generally, $?A \sqcup B = B \sqcup ?A$. Mappings preserving links must preserve $\lessdot$. The above definition is immediately extended to Quasi-Structures.

Definition. A *proof-structure* $\mathcal{S}$ for (*noncommutative*) first order **NCMALL** is defined as a proof-structure for commutative **MALL** together with a cyclic ordering $\lessdot$. It satisfies conditions (1), (2), (4) and in addition the following property:

(3)" Fam *preserves* $\lessdot$ *and every quasi-structure in* Fam *$(\mathcal{S}_\varphi)$ satisfies the planarity condition.*

The definition of an *inductive proof-structure* for **NCMALL**, sections (3.1) and (3.2), is similar to that for the commutative case; in addition, we inductively define $\lessdot$. Write $\Gamma = X_0, \ldots, X_{n-1}$, where for $i = 0, \ldots, n-1$ (mod n), $X_i \lessdot X_{i+1}$. In clause (2) if $\mathcal{S}'$ has conclusions $A_0, \ldots, A_{n-1}, A$ (mod $n+1$) and $\mathcal{S}''$ has conclusions $B, B_0 \ldots, B_{m-1}$ (mod $m+1$) then $\mathcal{S}'\,{}_A\!\otimes_B \mathcal{S}''$ has conclusions if $\mathcal{S}'$ has conclusions $C_0, \ldots, C_{n+m}$ (mod $n+m+1$) where $C_i = A_i$ for $i = 0, \ldots, n-1$, $C_n = A \otimes B$ and $C_{n+j+1} = B_j$ for $j = 0, \ldots, m-1$. If the new link introduced in clause (2) is a cut-link, then the conclusions of $\mathcal{S}'\,{}_A\!\otimes_{A^\perp} \mathcal{S}''$ are $C_0, \ldots, C_{n+m-1}$ (mod $n+m$) where $C_i = A_i$ for $i = 0, \ldots, n-1$ and $C_{n+j} = B_j$ for $j = 0, \ldots, m-1$. In clause (3) if $\mathcal{S}'$ has conclusions $C_0, \ldots, C_n$ (mod $n+1$) with $C_{n-1} = A$ and $C_n = B$, then $\mathcal{S}$ has conclusions $C_0, \ldots, C_{n-1}$, (mod n) with $C_{n-1} = A \sqcup B$. In clause (7), $\mathcal{S}'_{\varphi'}$, $\mathcal{S}''_{\varphi''}$ and $\mathcal{S}_\varphi$ have conclusions $C'_0, \ldots, C'_n$, $C''_0, \ldots, C''_n$ and $C_0, \ldots, C_n$ (mod $n+1$), respectively, where we require that $C_i \sim C'_i \sim C''_i$ for $i = 0, \ldots, n-1$ and $C_n = C'_n \sqcap C''_n$. In all other clauses the ordering is also preserved.

The definition of empire is unchanged. Proof-nets are defined as before, with the new notion of proof-structure. Since all the fundamental notions are independent of the restrictions induced by $\lessdot$ on a proof-structure, the Main Theorem in section (4) and the Equivalence Theorem (section 6.3) carry through to the non-commutative case. In particular, the map $\pi : \mathcal{D} \to \mathcal{S}_\mathcal{D}$ between derivations and proof-structures for **NCMALL** is defined as in section (3.7). The Tiling Lemma and Substructure Theorem, sections (4.3) and (4.5), still hold. The sequentialization theorem also holds, as we can see from the following argument.

Proof of the sequentialization theorem for NCMLL$^-$. (see section 4.4.1). In the difficult case, given a proof-net $\mathcal{S}$ with conclusions $C_0, \ldots, C_n$ (mod $n+1$), we break the final link $\frac{A \quad B}{A \sqcap B}$ of a conclusion C_i and we partition $C_0,\ldots,C_{i-1}$, A, B, C_{i+1}, …, C_n into two classes. By applying a cyclic permutation to the conclusions, if necessary, we may assume that C_0 is in the class which A belongs to, and that $C_{i+1},\ldots,C_n$ are not. Therefore, $C_{i+1},\ldots,C_n$ are in the same class as B.

Suppose now that some C_j is in the same class as B, for $0 < j < i$, Then by the connectedness condition there must be C_k, with $0 < k < i$, such that C_k is connected to one of B, $C_{i+1},\ldots,C_n$, by an axiom $\overline{P, P^{\perp}}$, say $P \prec C_k$ and $P^{\perp} \prec B$. For the same reason there must be some D and E in the same class as A, such that $D \lessdot C_k \lessdot E$ and D and E are connected by an axiom $\overline{Q, Q^{\perp}}$, say $Q \prec D$ and $Q^{\perp} \prec E$. We are in the following situation:

$$C_0 \ldots, D, \ldots, C_k, \ldots, E, \ldots, A, B, C_{i+1}, \ldots, C_n.$$

In conclusion in $\mathcal{S}$ we have axioms $\overline{P, P^{\perp}}$ and $\overline{Q, Q^{\perp}}$ such that

$$Q \lessdot P \lessdot Q^{\perp} \lessdot P^{\perp}.$$

This violates condition (3)'. Therefore $C_0,\ldots,C_{i-1},A$ all belong to the same class, and the splitting is compatible with the restrictions on the noncommutative case. ∎

7. Cut Elimination.

Let $\mathcal{S}_\varphi(\Gamma)$ be a proof-net. A Cut link, as defined in section (2.1), has a ghost formula-occurrence of the form $\underline{A \otimes A^\perp}$ as a conclusion. Since $\underline{A \otimes A^\perp}$ is not a formula occurrence, it is neither a premise of any link nor a conclusion of $\mathcal{S}_\varphi$.

Definition. Let **CUT** be the set of ghost formula-occurrences in $\mathcal{S}_\varphi$ and let $\underline{A \otimes A^\perp} \in$ **CUT**. A *cut-reduction* for the proof-structure $\mathcal{S}_\varphi$ is the replacement of $\mathcal{S}_\varphi(\Gamma)$ with a proof-structure $\mathcal{S}'_{\varphi'}(\Gamma)$ with a set **CUT'** of cuts, such that $\underline{A \otimes A^\perp}$ does not occur in **CUT'**, but one or two new ghost formula-occurrences of lower logical complexity may occur in **CUT'**.

A ghost formula occurrence $\underline{A \otimes A^\perp}$ in $\mathcal{S}_\varphi(\Gamma)$ is mapped under *Fam* to an equivalence class $[\underline{A \otimes A^\perp}]$ in $\mathcal{F} = \mathit{Fam}\ (\mathcal{S}_\varphi)$. If the proof-structure $\mathcal{S}_\varphi(\Gamma)$ reduces to $\mathcal{S}'_{\varphi'}(\Gamma)$, then the result of a cut-reduction applied to the family *Fam* $(\mathcal{S}_\varphi(\Gamma))$ is defined as the family *Fam* $(\mathcal{S}'_{\varphi'}(\Gamma))$.

There are as many cut reductions as there are pairs of links for dual logical symbols: $\mathbf{1}/\perp$, $\otimes/\sqcup$, $\sqcap/\oplus$, $\bigwedge / \bigvee$. Since there is no rule nor axiom for $\mathbf{0}$, there is no $\top/\mathbf{0}$ reduction. There is a *commutative* $\top$ *reduction* by which a $\top$-axiom 'absorbs' any proof-structure $\mathcal{S}$ to which it is connected by a $\otimes$-link. In addition, there are commutative $\sqcap$-reductions for proof-nets. The formalism of proof-networks has been designed in order to avoid commutative $\sqcap$-reductions: indeed, if $\mathcal{S}_\varphi$ results from $\mathcal{S}'_{\varphi'}$ by such a reduction, then *Fam* $(\mathcal{S}_\varphi)$ and *Fam* $(\mathcal{S}'_{\varphi'})$ are the same *set of quasi-structures* (of course, since φ and φ' are different, the 'box condition' applies to different classes of formulas in *Fam* $(\mathcal{S}_\varphi)$ and *Fam* $(\mathcal{S}'_{\varphi'})$.

It is clear that for proof-networks Normalization and Strong Normalization coincide and that the Cut-free proof-networks obtained by any two sequences of reductions coincide.

It is therefore enough to show that if *Fam* $(\mathcal{S}_\varphi)$ is a proof-network then so is the family *Fam* $(\geq'_{\varphi'})$ resulting from a cut reduction (*Reduction Lemma*). We check the vicious circle and connectedness condition separately for each $\mathcal{Q} \in$ *Fam* $(\mathcal{S}_\varphi)$; the proof is simplified by the use of the *method of chains* (section 5). However, it is convenient to verify the parameters condition in the corresponding proof-nets, using the Equivalence Theorem (section 6.3).

7.1. Cut Reductions for Proof-Nets.

The following are the cut-reductions available for proof-nets for **NCMALL** and **MALL**.

• **Axiom Reductions**

$$\begin{array}{cc} \overline{X_1, \ldots, X_m, \quad P} & \overline{P^\perp \quad Y_1, \ldots, Y_n} \\ \multicolumn{2}{c}{\underline{P \otimes P^\perp}} \end{array}$$

reduces to

$$\overline{X_1, \ldots, X_m, Y_1, \ldots, Y_n}$$

• **1 Reduction**

$$\begin{array}{cc} \overline{X_1, \ldots, X_n \quad \perp} & \overline{\mathbf{1} \quad \perp_1, \ldots, \perp_m,} \\ \multicolumn{2}{c}{\underline{\perp \otimes \mathbf{1}}} \end{array}$$

reduces to

$$\overline{X_1, \ldots, X_n, \perp_1, \ldots, \perp_m}$$

• *Times* **Reduction.**

$$\dfrac{\dfrac{A \quad B}{A \otimes B} \qquad\qquad \dfrac{B^\perp \quad A^\perp}{B^\perp \sqcup A^\perp}}{\underline{(A \otimes B) \otimes (A \otimes B)^\perp}}$$

reduces to

$$\underline{\dfrac{A \quad A^\perp}{A \otimes A^\perp}} \qquad \underline{\dfrac{B^\perp \quad B}{B^\perp \otimes B}}$$

• *With* **Reduction.**

$$\begin{array}{ccccc} & \mathcal{S}_1 \qquad \mathcal{S}_2 & & & \\ \dfrac{X_1' \quad X_1''}{X_1} & \cdots & \dfrac{X_n' \quad X_n''}{X_n} & \dfrac{A_1 \quad A_2}{A_1 \sqcap A_2} & \dfrac{A_i^\perp}{A_1^\perp \oplus A_2^\perp} \\ \multicolumn{3}{c}{\cdots\cdots\cdots\cdots\cdots\cdots} & \multicolumn{2}{c}{\underline{(A_1 \sqcap A_2) \otimes (A_1 \sqcap A_2)^\perp}} \end{array}$$

where $\mathcal{S}_1 = e(A_1)$ and $X_1', \ldots, X_n'$, A_1 are all the doors of $e(A_1)$, and $\mathcal{S}_2 = e(A_2)$ and $X_1'', \ldots, X_n''$, A_2 are all the doors of $e(A_2)$, reduces to

$$\begin{array}{ccc} & \mathcal{S}_i & \\ X_1 \ldots X_n & \dfrac{A_i \qquad A_i^{\perp}}{A_i \otimes A_i^{\perp}} \\ \ldots\ldots & & \end{array}$$

• $\bigwedge$ **Reduction**

$$\begin{array}{c} e(A(a)) \\ \dfrac{\dfrac{A(a)}{\bigwedge x.A} \qquad \dfrac{A(t)^{\perp}}{\bigvee x.A^{\perp}}}{\underline{\bigwedge x.A \otimes (\bigwedge x.A)^{\perp}}} \end{array}$$

reduces to

$$\begin{array}{c} e(A(t)) \\ \dfrac{A(t) \qquad A(t)^{\perp}}{\underline{A(t) \otimes A(t)^{\perp}}} \end{array}$$

• **Commutative** $\top$ **Reduction.**

$$\overline{\top, X_1, \ldots, X_m, \quad A} \qquad \dfrac{A \qquad A^{\perp}}{\underline{A \otimes A^{\perp}}} \quad \begin{array}{c} e(A^{\perp}) \\ Y_1, \ldots, Y_n \end{array}$$

where $Y_1, \ldots, Y_n$ are all the doors of $e(A^{\perp})$, reduces to

$$\overline{\top, X_1, \ldots, X_m, Y_1, \ldots, Y_n}$$

• **Commutative *With* Reduction.**

$$\begin{array}{cccccc} & \mathcal{S}_1 \quad \mathcal{S}_2 & & & & \\ \dfrac{A_1 \quad A_2}{A_1 \sqcap A_2} & \dfrac{X_1' \quad X_1''}{X_1} & \ldots & \dfrac{X_n' \quad X_n''}{X_n} & X_n^{\perp} & \begin{array}{c} e(X_n^{\perp}) \\ Y_1, \ldots, Y_m \end{array} \\ \ldots\ldots\ldots\ldots & & & \multicolumn{2}{c}{\underline{Y_n \otimes X_n^{\perp}}} & \ldots\ldots \end{array}$$

where $\mathcal{S}_1 = e(A_1)$ and $X'_1,\ldots,X'_n$, A_1 are all the doors of $e(A_1)$, $\mathcal{S}_2 = e(A_2)$ and $X''_1,\ldots,X''_n$, A_2 are all the doors of $e(A_2)$ and X_n, $Y_1,\ldots,Y_m$ are all the doors of $e(X_n)$, reduces to

$$
\begin{array}{cccccccc}
\multicolumn{4}{l}{\mathcal{S}_1 \ {}_{X'_n}\!\otimes_{X'^{\perp}_n} e(X'^{\perp}_n)} & \multicolumn{4}{l}{\mathcal{S}_2 \ {}_{X''_n}\!\otimes_{X''^{\perp}_n} e(X''^{\perp}_n)} \\
\underline{A_1 \quad A_2} & \underline{X'_1 \quad X''_1} & \cdots & \underline{X'_n \ X'^{\perp}_n} & \underline{X''_n \ X''^{\perp}_n} & \underline{Y'_1 \quad Y''_1} & \cdots & \underline{Y'_m \quad Y''_m} \\
A_1 \sqcap A_2 & X_1 & & \underline{X'_n \otimes X'^{\perp}_n} & \underline{X''_n \otimes X''^{\perp}_n} & Y_1 & & Y_m \\
\multicolumn{3}{c}{\cdots\cdots\cdots\cdots\cdots} & & & \multicolumn{3}{c}{\cdots\cdots\cdots\cdots\cdots}
\end{array}
$$

where A, $X'_1,\ldots,X'_{n-1}$, $Y'_1,\ldots,Y'_m$ are all the formula-occurrences flagged with A and B, $X''_1,\ldots,X''_{n-1}$, $Y''_1,\ldots,Y''_m$ are all the formula-occurrences flagged with B. A similar reduction is defined for a cut-link with conclusion $\underline{X_i \otimes X_i^{\perp}}$, for $i < n$.

Remarks. (i) The axiom reduction presupposes that axioms consist of atomic formulas. In the most general case, a cut link occurrs in a structure $\mathcal{S}_1 \ {}_A\!\otimes_{A^{\perp}} \mathcal{S}^2$ where say $\mathcal{S}_1$ is an axiom and $\mathcal{S}_2$ is not. When such a reduction is performed it is necessary to introduce some occurrences of $\perp$ in the axioms of $\mathcal{S}_2$. This is a process involving arbitrary choices.

(ii) In the first order case, we assume that *ghost formula-occurrences are the only formulas in the conclusion in which eigenvariables may occur.*

7.2. Remarks on Reductions for Proof-Networks.

Remark 1. A *With* reduction is as follows. For every $\mathcal{Q} \in \mathcal{F}$ such that $\mathcal{Q} \cap [\underline{(A_1 \sqcap A_2) \otimes (A_1 \sqcap A_2)^{\perp}}] \neq \emptyset$, $\mathcal{Q}$ contains either a configuration of the form (1):

$$
\begin{array}{cc}
\mathcal{Q}_i & \\
\underline{A_i} & \underline{A_i^{\perp}} \\
A_1 \sqcap A_2 & A_1^{\perp} \oplus A_2^{\perp} \\
\hline
\multicolumn{2}{c}{\underline{(A_1 \sqcap A_2) \otimes (A_1 \sqcap A_2)^{\perp}}}
\end{array}
$$

or of the form (2):

$$\dfrac{\dfrac{\begin{matrix}\mathcal{Q}_i\\ A_i\end{matrix}}{A_1 \sqcap A_2} \qquad \dfrac{A_j^{\perp}}{A_1^{\perp} \oplus A_2^{\perp}}}{\underline{(A_1 \sqcap A_2) \otimes (A_1 \sqcap A_2)^{\perp}}}$$

where $i \neq j$. Now $\mathcal{F}'$ is obtained from $\mathcal{F}$ as follows: In the case (1) we replace $\mathcal{Q}$ with a $\mathcal{Q}'$ of the form (3):

$$\begin{matrix}\mathcal{Q}_i & \\ \dfrac{A_i \qquad A_i^{\perp}}{\underline{A_i \otimes A_i^{\perp}}}\end{matrix}$$

In the case (2), we delete $\mathcal{Q}$.

Remark 2. A *Times* reduction determines the following situation.

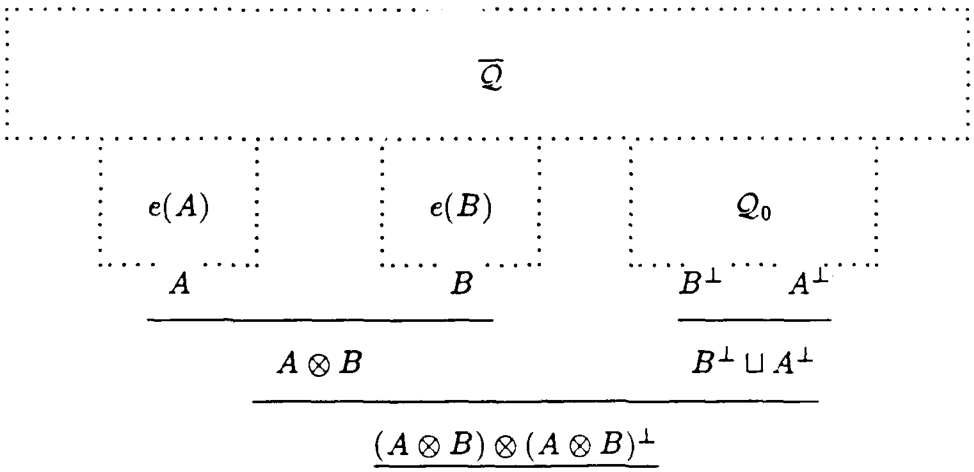


Figure 1. A tailing of $\mathcal{Q}$.

In figure 1 $\mathcal{Q}_0$ is $e(B^{\perp}) = e(A^{\perp})$ and $\overline{\mathcal{Q}}$ (corresponds to) the complementary substructure of

$$\dfrac{\dfrac{e(A) \quad e(B)}{A \otimes B} \qquad \dfrac{\mathcal{Q}_0}{(A \otimes B)^{\perp}}}{(A \otimes B) \otimes (A \otimes B)^{\perp}} \; .$$

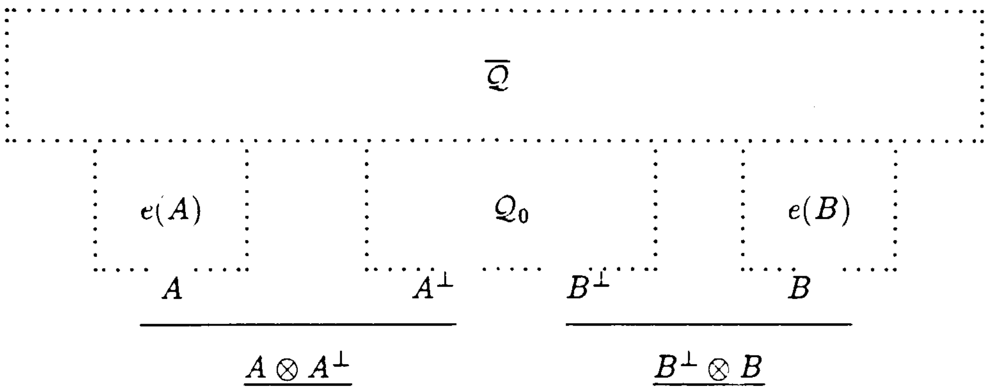


Figure 2. A tailing of $\mathcal{Q}'$.

In figure 2 we have $\mathcal{Q}_0 = e(B^\perp) \cap e(A^\perp)$: this follows from the Shared Empires Lemma (section 4.3, extended to the **MALL** case) and the fact that the reduction step does not affect the links in $\mathcal{Q}_0$ and between $\mathcal{Q}_0$ and $\overline{\mathcal{Q}}$. It follows also that if X, Y occur in $\mathcal{Q}_0$, then $Y \in e(X)$ in $\mathcal{Q}$ if and only if $Y \in e(X)$ in $\mathcal{Q}'$.

Remark 3. There are obvious embeddings among the substructures obtained by removing all links with premises A, $A^\perp$, B, $B^\perp$ in figures 1 and 2. By an abuse of language, we sometimes identify copies of the corresponding formula-occurrences. A similar remark applies to figures 3 and 4 below. By a further abuse of notation, if $X_{\mathcal{Q}}$ is the representative of $[X]$ in $\mathcal{Q}$, we sometimes omit the subscript $\mathcal{Q}$, assuming it is clear from the context.

7.3. Strong Cut Elimination.

Strong Cut Elimination for first order **MALL.** *Given $\mathcal{F} = Fam\ (\mathcal{S}_\varphi)$, every reduction sequence terminates in a cut-free $\mathcal{F}_0 = Fam\ (\mathcal{S}^0_{\varphi_0})$ and $\mathcal{F}_0$, regarded as a set of quasi-structures, is unique.* .

We must prove the following:

Reduction Lemma. *If $\mathcal{F} = Fam\ (\mathcal{S}_\varphi)(\Gamma))$ is a proof-network, and $\mathcal{S}_\varphi(\Gamma)$ reduces to $\mathcal{S}'_{\varphi'}(\Gamma)$ by application of a cut-reduction, then $\mathcal{F}' = Fam\ (\mathcal{S}'_{\varphi'})(\Gamma))$ is a proof-network.*

It is clear that every reduction step reduces the total number of formula occurrences in the proof-network. We work with finite proof-networks, thus every reduction sequence terminates. It is also clear that normal forms are unique in the sense specified by the theorem. Thus to prove the Strong Cut Elimination Theorem for **MALL** we need only prove the Reduction Lemma. We will use extensively the notions of path and chain developed in section (5).

Let $\mathcal{F}$ be a proof-network and let $\mathcal{F}'$ be the result of one reduction step. To prove that $\mathcal{F}'$ satisfies the *connectedness* condition we show that if for any $\mathcal{Q}' \in \mathcal{F}'$ the set of axioms of $\mathcal{Q}'$ is a path which is not minimal, then there is a $\mathcal{Q} \in \mathcal{F}$ such that the set of axioms of $\mathcal{Q}$ is a non-minimal path. To prove that $\mathcal{F}'$ satisfies the *vicious circle* condition we show that for any $\mathcal{Q}' \in \mathcal{F}'$, if there is a cycle in $\mathcal{Q}'$ then there is one also in some $\mathcal{Q} \in \mathcal{F}$.

Proof of the Reduction Lemma. *Commutative* $\top$*-reduction* Consider the complementary substructure of $e(A^{\perp})$ in the given structure and apply the Substructure Theorem section (4.5). The result is then immediate.

Commutative $\sqcap$*-reduction.* We need to check that if $\mathcal{S}'_{\varphi'}$ is the result of the reduction, then A, $X'_1,\ldots,X'_{n-1}$, $Y'_1,\ldots,Y'_m$ are all the doors of $e(A)$ and B, $X''_1,\ldots,X''_{n-1}$, $Y''_1,\ldots,Y''_m$ are all the doors of $e(B)$ in $\mathcal{S}'_{\varphi'}$. In other words, we need to show that

$$e(A_1) = \mathcal{S}_1\ {}_{X'_n \otimes X'^{\perp}_n}\ e(X'^{\perp}_n)$$

and

$$e(A_2) = \mathcal{S}_2\ {}_{X''_n \otimes X''^{\perp}_n}\ e(X''^{\perp}_n).$$

The definition of $\mathcal{S}_i$ and easy inductions on $e(X'^{\perp})$ and $e(X''^{\perp})$ show that the right-hand side is included in the lest-hand s[illegible] in both cases. A standa[illegible] argument contradicts the existence of a normal tree $\tau^{\omega}[A_1]$ such that Z is the first formula-occurrence not in $\mathcal{S}_1\ {}_{X'_n \otimes X'^{\perp}_n}\ e(X'^{\perp}_n)$.

Axiom or **1 reduction.** Let $\mathcal{Q} \in \mathcal{F}$ and let $\mathcal{Q}' \in \mathcal{F}'$ result from $\mathcal{Q}$ by an axiom or 1 reduction (section 7.1). Let $\mathcal{P}$ $[\mathcal{P}']$ be a list of all axioms occurring in $\mathcal{Q}$ $[\mathcal{Q}']$. Also let $\mathcal{A}_1$ and $\mathcal{A}_2$ be the indicated axioms of $\mathcal{Q}$ and let $\mathcal{A} = \overline{X_1,\ldots,X_m,Y_1,\ldots,Y_n}$ be the axiom resulting from the reduction.

Connectedness. Suppose $\mathcal{P}''$ is a proper subset of $\mathcal{P}'$ and a path for $\mathcal{Q}'$. If $\mathcal{P}''$ does not contain $\mathcal{A}$ then $\mathcal{P}''$ is a path for $\mathcal{Q}$, too; otherwise the path $\mathcal{P}'''$ obtained by replacing $\mathcal{A}_1$ and $\mathcal{A}_2$ for $\mathcal{A}$ in $\mathcal{P}''$ is a path for $\mathcal{Q}$ and this is a contradiction.

Vicious circle. Let $\mathcal{C}$ be

$$C \otimes D - C_1 \otimes D_1 - \ldots - C_i \otimes D_i - C_{i+1} \otimes D_{i+1} - \ldots - C_n \otimes D_n - C \otimes D$$

be a cycle in $\mathcal{Q}'$. We assume that $\mathcal{A}$ is used in $\mathcal{C}$: otherwise, $\mathcal{C}$ is a chain in $\mathcal{Q}$ too. We may also assume that an axiom in $\mathcal{P}'$ is used at most once in $\mathcal{C}$: if, say, $X_i \prec D$, $Y_j \prec C_1$ and $X_k \prec D_i$, $Y_l \prec C_{i+1}$, then we could consider the shorter cycle $C \otimes D - C_{i+1} \otimes D_{i+1} - \ldots - C_n \otimes D_n - C \otimes D$. Therefore suppose that $X_i \prec D_i$ and $Y_j \prec C_{i+1}$. Then

$$C \otimes D - C_1 \otimes D_1 - \ldots - C_i \otimes D_i - \underline{P \otimes P\bot} - C_{i+1} \otimes D_{i+1} - \ldots - C_n \otimes D_n - C \otimes D$$

or

$$C \otimes D - C_1 \otimes D_1 - \ldots - C_i \otimes D_i - \underline{\bot \otimes \mathbf{1}} - C_{i+1} \otimes D_{i+1} - \ldots - C_n \otimes D_n - C \otimes D$$

is a cycle in $\mathcal{Q}$.

Box and Parameters Conditions. For all formula occurrences X, Y in $\mathcal{Q}'$, $Y \in e(X)$ if and only if for the corresponding X, Y in $\mathcal{Q}$ we have $Y \in e(X)$. It follows that $[Y] \in e[X]$ in $\mathcal{F}'$ if and only if $[Y] \in e[X]$ in $\mathcal{F}$. The box and parameters Conditions follow.

Times **reduction.** Every quasi-structure $\mathcal{Q}' \in \mathcal{F}'$ resulting from the reduction step has the form shown in figure 2 (section 7.1) and results from a $\mathcal{Q} \in \mathcal{F}$ of the form shown in figure 1.

Connectedness. Let $\mathcal{P}$ be a list of all axioms in $\mathcal{Q}'$. The elements of $\mathcal{P}$ are also (copies of) the axioms of $\mathcal{Q}$. Suppose $\mathcal{P}'$ is a proper subset of $\mathcal{P}$ which is a path for $\mathcal{Q}'$. We need only to consider the case when $\mathcal{P}' \not\mapsto A^\bot$ or $\mathcal{P}' \not\mapsto B^\bot$. Notice that since $\mathcal{P}'$ satisfies the relevance condition on conjuncts, $\mathcal{P}' \not\mapsto A$ implies $\mathcal{P}' \not\mapsto A^\bot$ and similarly for B. By the connectedness condition for $\mathcal{Q}$, applied to $(A \otimes B)^\bot = A^\bot \sqcup B^\bot$, we have $A^\bot \Diamond B^\bot$ in $(S^\prec, \mathcal{P})$ and obviously this fact remains true in $\mathcal{Q}'$.

By part (i) of the Proposition in Section (5.3), $\mathcal{P}' \not\rightarrow A^\perp$ and $\mathcal{P}' \not\rightarrow B^\perp$. It follows that $\mathcal{P}' \not\rightarrow (A \otimes B) \otimes (A \otimes B)^\perp$ in $\mathcal{Q}$. But $e((A \otimes B) \otimes (A \otimes B)^\perp) = \mathcal{Q}$, by connectedness for $\mathcal{Q}$ and therefore $\mathcal{P}'$ is empty, a contradiction.

Vicious Circle. Suppose the vicious circle condition is violated in $\mathcal{Q}'$. We may assume we have a cycle $\mathcal{C}'$ containing $\underline{A \otimes A^\perp}$ or $\underline{B \otimes B^\perp}$:

$$C \otimes D - C_1 \otimes D_1 - \ldots - C_i \otimes D_i - \ldots - C_j \otimes D_j - - \ldots - C_n \otimes D_n - C \otimes D.$$

and we show that there is a cycle $\mathcal{C}$ in $\mathcal{Q}$. (Here we consider only the case of Commutative Linear Logic.) There are several cases. We use the same letters for formula-occurrences identified by the embeddings described in Remark 3, section 7.2.

Suppose $C_i = A$ and $D_i = A^\perp$ and $\underline{B \otimes B^\perp}$ does not occur in $\mathcal{C}'$. Then $\mathcal{C}$ is obtained from $\mathcal{C}'$ by substituting $\underline{(A \otimes B) \otimes (A \otimes B)^\perp}$ for $C_i \otimes D_i$. Similarly if $C_i = B$ and $D_i = B^\perp$ and $\underline{A \otimes A^\perp}$ does not occur in $\mathcal{C}'$.

Suppose $C_i = A$, $D_i = A^\perp$ and $C_j = B$, $D_j = B^\perp$. Then let $\mathcal{C}$ be

$$C \otimes D - \ldots - C_{i-1} \otimes D_{i-1} - \underline{(A \otimes B) \otimes (A \otimes B)^\perp} - C_{j+1} \otimes D_{j+1} - \ldots - C \otimes D.$$

Suppose $C_i = A$, $D_i = A^\perp$ and $C_j = B^\perp$, $D_j = B$. Then let $\mathcal{C}$ be

$$C \otimes D - \ldots - C_{i-1} \otimes D_{i-1} - A \otimes B - C_{j+1} \otimes D_{j+1} - \ldots - C \otimes D.$$

Suppose $C_i = A^\perp$, $D_i = A$ and $C_j = B^\perp$, $D_j = B$. Then let $\mathcal{C}$ be

$$C \otimes D - \ldots - C_{i-1} \otimes D_{i-1} -_{(A \otimes B)^\perp} \underline{(A \otimes B) \otimes (A \otimes B)^\perp}_{(A \otimes B)} - C_{j+1} \otimes D_{j+1} - \ldots - C \otimes D.$$

Finally, suppose $C_i = A^\perp$, $D_i = A$ and $C_j = B$, $D_j = B^\perp$. Then let $\mathcal{C}$ be

$$A \otimes B_A - C_{i+1} \otimes D_{i+1} - \ldots -_B A \otimes B.$$

In all cases we have found a cycle in $\mathcal{Q}$, a contradiction.

Box Condition. Given $\mathcal{F} = Fam\ (S_\varphi)$ and $\mathcal{F}' = Fam\ (S'_{\varphi'})$, notice that the assignments of φ and φ' are the same (modulo the embeddings described in Remark 3, section 7.2).

Consider figures 1 and 2, section (7.2). If $X_{\mathcal{Q}}$ occurs either (1) in $\overline{\mathcal{Q}}$ or (2) in $e(A)$ or (3) in $e(B)$ or (4) in $\mathcal{Q}_0$ and $A^\perp, B^\perp \in e(X_{\mathcal{Q}})$, then by the Tiling Lemma the corresponding formula occurrence $X_{\mathcal{Q}'}$ has the property that $\underline{A \otimes A^\perp} \in e(X_{\mathcal{Q}'})$, $B \otimes B^\perp \in e(X_{\mathcal{Q}'})$. It follows that for every Z different from $\underline{A \otimes A^\perp}$ and $\underline{B \otimes B^\perp}$, $Z_{\mathcal{Q}'} \in e(X_{\mathcal{Q}'})$ if and only if $Z_{\mathcal{Q}} \in e(X_{\mathcal{Q}})$. In this case we conclude that $[Z] \in e[X]$ *in* $\mathcal{F}'$ if and only if $[Z] \in e[X]$ *in* $\mathcal{F}$.

Otherwise, suppose that a representative $(X \sqcap Y)_{\mathcal{Q}}$ of $[X \sqcap Y]$ in $\mathcal{Q}$ occurs in $\mathcal{Q}_0$ let Z be any formula occurrence flagged with X or Y, say $\varphi(Z) = X$. We have $Z_{\mathcal{Q}} \in e(X_{\mathcal{Q}})$ by the box condition for $\mathcal{F}$. If $Z_{\mathcal{Q}}$ does not occur in $e(A)$ or $e(B)$, then it is clear that $Z_{\mathcal{Q}'}$ still belongs to $e(X_{\mathcal{Q}'})$. So assume $Z_{\mathcal{Q}} \in e(A)$. By the vicious circle condition, $e(A) \cap e(A^\perp) = \emptyset$, hence $X_{\mathcal{Q}} \notin e(A)$. By the Tiling Lemma, $A \sqsubset X_{\mathcal{Q}}$ and it follows that $A \otimes B \in e(X_{\mathcal{Q}})$, $B^\perp \sqcup A^\perp \in e(X_{\mathcal{Q}})$ hence $A^\perp, B^\perp \in e(X_{\mathcal{Q}}$, contradicting the assumption of the case.

Parameters Condition. We need to check that the ordering $<_t$ of the eigenvariables is strict in $\mathcal{F}'$, assuming that it is such in $\mathcal{F}$. Let $\bigwedge x.C$ be a formula occurrence in $\mathcal{Q}$ with associated eigenveriable c. We say that c is *critical* if $C(c)$ (and so $\bigwedge x.C$) occurs in $\mathcal{S}_0$ and only one of $A^\perp$ and $B^\perp$ belongs to $e(C(c))$.

By the first paragraph in the argument for the box condition, if c is not critical, then for all X different from $\underline{(A \otimes B) \otimes (A \otimes B)^\perp}$ $(A \otimes B)$ and $(B^\perp \sqcup A^\perp)$, we have that $X_{\mathcal{Q}} \in e(C(c))$ if and only if $X_{\mathcal{Q}'} \in e(C(c))$, hence $[X] \in e[C(c)]$ in $\mathcal{F}$ if and only if $[X] \in e[C(c)]$ in $\mathcal{F}'$. Furthermore, if any eigenvariable c occurs in $\underline{(A \otimes B) \otimes (A \otimes B)^\perp}$, $(A \otimes B)$ and $(B^\perp \sqcup A^\perp)$, then it occurs also somewhere in $\mathcal{S}_0$ and in $e(A)$ or $e(B)$.

Therefore it is sufficient to consider a sequence $a_0 <^0 a_1 <^0 \ldots <^0 c <^0 a_i \ldots <^0 a_n$ in $\mathcal{F}'$ where c is critical and it is sufficient to show that we do not have $c <_t c$.

For all $\mathcal{Q}''$, write $\mathcal{Q}''_0$ for $e(A^\perp) \cap e(B^\perp)$ in $\mathcal{Q}''$. By the Shared Empires Lemma $\mathcal{Q}_0$ is (isomorphic to) $\mathcal{Q}'_0$. We write therefore $\mathcal{F}_0$ for the family $\{\mathcal{Q}''_0 : \mathcal{Q}'' \in \mathcal{F}\}$, which coincides with the family $\{\mathcal{Q}''_0 : \mathcal{Q}'' \in \mathcal{F}'\}$.

Since $\mathcal{F}$ is a proof-network, by the Equivalence Theorem (section 6.3) there is a proof-net $\mathcal{S}_\varphi$. and by the Substructure Theorem (Corollary, section 4.5) the substructure $\mathcal{S}_{0_{\varphi_0}}$ is a subnet. By part (*ii*) of the proposition in section (4.6) the ordering $<_t$ in $\mathcal{S}_{0_{\varphi_0}}$ is the restriction of the ordering $<_t$ to $\mathcal{S}_{0_{\varphi_0}}$.

We claim that for all eigenvariables b such that $b <_t c$ in $\mathcal{F}'$ and for all $\mathcal{Q}'' \in \mathcal{F}'$, b occurs only inside $e(C(c)) \cap \mathcal{Q}_0''$ and therefore $b <_t c$ already in $\mathcal{F}$. From this we conclude that $c <_t c$ is impossible in $\mathcal{F}'$ since $c <_t c$ is impossible in $\mathcal{F}$.

The proof of the claim is by induction on the length of the sequence $c = a_0 >^0 a_1 >^0 \ldots >^0 a_n = b$ *in* $\mathcal{F}'$, where a_i is the eigenvariable associated with $\bigwedge x_i.A_i$. Since by induction hypothesis a_i occurs only inside $e[C(c)] \cap [\mathcal{Q}_0]$, and a_i occurs outside $e[A_{i+1}]$, we cannot have $e[C(c)] \cap [\mathcal{Q}_0] \subset e[A_{i+1}]$, in particular, $\bigwedge x_{i+1}.A_{i+1}$ must occur in $\mathcal{Q}_0$.

Therefore we can argue in the corresponding proof-structure $\mathcal{S}_0$. Since $a_i >^0 a_{i+1}$, a^i occurs inside $e(A_{i+1}(a_{i+1})$; by induction, a_i occurs only in $e(C(c))$. Thus $e(C(c)) \cap e(A_{i+1}(a_{i+1})) \neq \emptyset$ and not $C(c) \sqsubset A_{i+1}(a_{i+1})$ implies $C(c) \Diamond A_{i+1}(a_{i+1})$ or $A_{i+1}(a_{i+1}) \sqsubset C(c)$ and from this it follows that a_{i+1} occurs inside $e(C(c))$.

Suppose a_{i+1} occurs outside $e[C(c)] \cap [\mathcal{Q}_0]$. Since in $\mathcal{F}$ we have $e[C(c)] = e[C(c)] \cap [\mathcal{Q}_0]$, it follow that $a_{i+1} >^0 c$, a contradiction.

With **reduction.** Let $\mathcal{F}'$ be obtained from $\mathcal{F}$ by eliminating a ghost formula-occurrence $\underline{(A_1 \sqcap A_2) \otimes (A_1 \sqcap A_2)^\perp}$.

The vicious circle and connectedness conditions are obviously satisfied by every $\mathcal{Q}' \in \mathcal{F}'$, since the transformation of a $\mathcal{Q} \in \mathcal{F}$ into $\mathcal{Q}' \in \mathcal{F}'$ in this representation involves only unary links. The box and parameters conditions for $\mathcal{F}'$ are immediate from the parameters condition for $\mathcal{F}'$.

$\bigwedge$ **reduction.** Since the $\bigwedge$ links and the $\bigvee$ links are both unary, the vicious circle, connectedness and box conditions are immediate. In order to prove that the parameters condition is also satisfied in this case it is convenient to apply the Equivalence Theorem and work with proof-structures instead.

The application of a $\bigwedge$ reduction (section 7.1) produces the following situation (see figures 3 and 4).

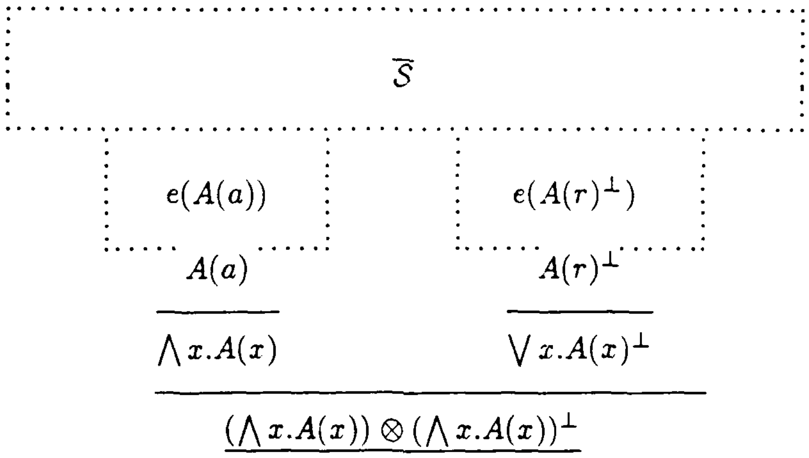


Figure 3. A tailing of $\mathcal{S}_{\varphi}$.

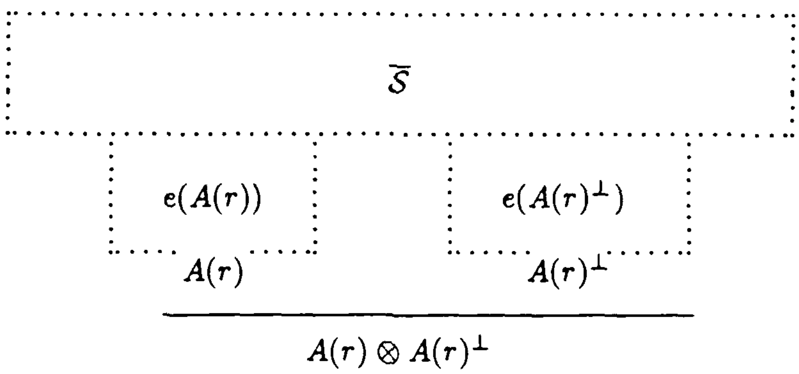


Figure 4. A tailing of $\mathcal{S}'_{\varphi'}$.

Remark. In figure 4 $e(A(r))$ is the result of replacing r for a everywhere in $e(A(a))$. Since a does not occur outside $e(A(a))$ by the parameters condition for $\mathcal{S}_{\varphi}$, it is immediate that $\overline{\mathcal{S}}$ is the same in the two figures. Also if X and Y are formula occurrences of $\mathcal{S}'_{\varphi'}$ different from $A(r) \otimes A(r)^{\perp}$ then there are formula occurrences X_0 and Y_0 in $\mathcal{S}_{\varphi}$ such that X and Y result from X_0, Y_0, respectively, by substituting every occurrence of a (if any) with r and $Y \in e(X)$ in $\mathcal{S}'$ if and only if $Y_0 \in e(X_0)$ in $\mathcal{S}$.

We need to check that if the ordering $<_t$ of the eigenvariables is strict in $\mathcal{S}'_{\varphi'}$, under the assumption that it is strict in $\mathcal{S}_\varphi$. If r does not contain any eigenvariables, then the result is immediate. Otherwise, it is enough to prove that if c is an eigenvariable occurring in r and b is any eigenvariable, then we cannot have $c <^0 b <_t c$ The proof depends on the following

Fact. *If $b <_t c$ in $\mathcal{S}'_{\varphi'}$, then either $b <_t c$ or $b <_t a$ in $\mathcal{S}_\varphi$.*

Work in $\mathcal{S}_\varphi$. Suppose r contains the eigenvariable c, where c is associated with $\bigwedge z.C$ and let b be any eigenvariable such that $b <_t a$, where b is associated with $\bigwedge y.B$. Since c occurs only within $e(C(c))$, $A(r)^\perp \in e(C(c))$, therefore also $A(a) \in e(C(c))$.

If $C(c)$ occurs inside $e(A(a))$, then $A(a) \Diamond C(c)$ and $a <^0 c$. Then $b <_t c$ and by the Proposition in section (4.6), b occurs only inside $e(C(c))$. If $C(c)$ does not occur inside $e(A(a))$, then $A(a) \sqsubset C(c)$ and $e(A(a)) \subset e(C(c))$. Again by the Proposition in section (4.6) b occurs only inside $e(A(a))$, therefore b occurs only inside $e(C(c))$.

Now consider $\mathcal{S}'_{\varphi'}$ and suppose $b_n <^0 \ldots <^0 b_0 = c$, where b_i is associated with $\bigwedge y_i.B_i$. Let b_{i+1} be the first eigenvariable such that the following holds *in* $\mathcal{S}_\varphi$: either $b_i <_t c_i$, for some c_i occurring in r, or $b_i <_t a$ but not $b_{i+1} <_t c_i$ for all c_i occurring in r nor $b_{i+1} <_t a$. By the above Remark it must be the case that in $\mathcal{S}'_{\varphi'}$ but not in $\mathcal{S}_\varphi$ we have $b_{i+1} <^0 c$ for some c occurring in r. We show that this is impossible. We must consider two cases.

Case (1): In $\mathcal{S}_\varphi$, a occurs only inside and c only outside $e(B_{i+1}(b_{i+1}))$. This implies $A(a) \in e(B_{i+1})$. But $\bigwedge y_{i+1}.B_{i+1} \neq \bigwedge x.A(x)$ and so $A(r)^\perp \in e(B_{i+1})$, a contradiction.

Case (2): In $\mathcal{S}_\varphi$, c occurs only inside and a only outside $e(B_{i+1}(b_{i+1}))$. This implies $A(r)^\perp \in e(B_{i+1})$. But $B_{i+1} \neq A(r)^\perp$ and so $A(a) \in e(B_{i+1})$, a contradiction. This concludes the proof of the Fact.

If for some c occurring in r we have $b <_t c$ in $\mathcal{S}'_{\varphi'}$, then by the Fact b occurs only inside $e(C(c))$ in $\mathcal{S}_\varphi$, and therefore also in $\mathcal{S}'_{\varphi'}$. Hence for no eigenvariable b we have $c <^0 b <_t c$ in $\mathcal{S}'_{\varphi'}$ and the proof is finished. ∎

8. An Extension.

The propositional Direct Logics **DL** and **DL**$^+$ are considered, namely the propositional logics formalized in the propositional sequent calculus for **MLL**$^-$ (i.e., propositional Multiplicative Linear Logic without constants **1** and $\perp$), with in addition either the rule of Mingle (**DL**) or the unrestricted rule of Weakening (**DL**$^+$). For these fragments we provide here a correct proof of the main theorem of Ketonen and Weyhrauch [1984]. Extensions to larger systems (first order or systems with additive connectives) are possible, using the techniques of the previous chapters. We are mainly interested in establishing the Structure Theorem and the Sequentialization Theorem. Cut-elimination results for proof-nets in **DL** and **DL**$^+$ are also possible, but we will not consider them here.

By the Main Theorem for **MLL**$^-$ (section 4) and by the form of the rule of Mingle we know that a proof-structure representing a derivation in **DL** may not satisfy the connectedness condition. It is also clear that in a proof-structure representing a derivation in **DL**$^+$ some formulas may occur that have no ancestor in an axiom-link.

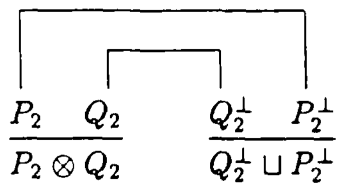
We will prove that *the condition* (1) there is no cycle *is necessary and sufficient to characterize proof-nets for* **DL** and that *the conditions* (1) there is no cycle *and* (2) the axioms form a minimal path *are necessary and sufficient to characterize proof-nets for* **DL**$^+$.

Consider the example (4) in section (3.5.1). The proof-structure $\mathcal{S}$ described there has the form

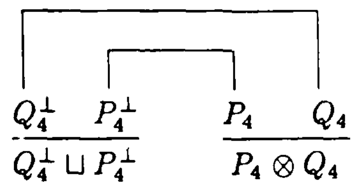
$$\mathcal{S} \;=\; \mathcal{S}_1 \;{}_{(Q_2^\perp \sqcup P_2^\perp)}\!\otimes_{(Q_1^\perp \sqcup P_1^\perp \sqcup R^\perp)}\; \mathcal{S}_2 \;{}_{(S^\perp \sqcup Q_3^\perp \sqcup P_3^\perp)}\!\otimes_{(Q_4^\perp \sqcup P_4^\perp)}\; \mathcal{S}_3, \qquad (\dagger)$$

where

$\mathcal{S}_1$:



$\mathcal{S}_3$:



and

$$\mathcal{S}_2:$$

$$\frac{P_1 \quad Q_1}{P_1 \otimes Q_1} \qquad \frac{\dfrac{Q_1^\perp \quad P_1^\perp}{Q_1^\perp \sqcup P_1^\perp} \quad R^\perp}{Q_1^\perp \sqcup P_1^\perp \sqcup R^\perp} \qquad \frac{R \qquad S}{R \otimes S} \qquad \frac{S^\perp \quad \dfrac{Q_3^\perp \quad P_3^\perp}{Q_3^\perp \sqcup P_3^\perp}}{S^\perp \sqcup Q_3^\perp \sqcup P_3^\perp} \qquad \frac{P_3 \quad Q_3}{P_3 \otimes Q_3}$$

Consider the conclusions of $\mathcal{S}$:

$$P_1 \otimes Q_1, P_2 \otimes Q_2, (Q_2^\perp \sqcup P_2^\perp) \otimes (Q_1^\perp \sqcup P_1^\perp \sqcup R^\perp),\ R \otimes S,$$
$$(S^\perp \sqcup Q_3^\perp \sqcup P_3^\perp) \otimes (Q_4^\perp \sqcup P_4^\perp),\ P_3 \otimes Q_3,\ P_4 \otimes Q_4.$$

It is clear that no formula occurrence in the conclusion contains all others in its empire — e.g., if U is $P_2 \otimes Q_2$ and V is $P_4 \otimes Q_4$, then $U \notin e(V)$ and $V \notin e(U)$, etc. Therefore no conclusion $X \otimes Y$ has the *splitting property*:

$$\mathcal{S} = e(X) \cup e(Y) \cup \{X \otimes Y\}.$$

Let $\mathcal{S}'$ be a proof-structure with the same conclusions as $\mathcal{S}$, but such that its axioms form a *minimal* path. For instance,

$$\mathcal{S}' = \mathcal{S}_1 \ {}_{(Q_2^\perp \sqcup P_2^\perp) \otimes (Q_1^\perp \sqcup P_1^\perp \sqcup R^\perp)}\ \mathcal{S}_4\ {}_{(S^\perp \sqcup Q_3^\perp \sqcup P_3^\perp) \otimes (Q_4^\perp \sqcup P_4^\perp)}\ \mathcal{S}_3$$

where

$$\mathcal{S}_4:$$

$$\frac{P_1^\perp \sqcup Q_1^\perp \quad R^\perp}{P_1^\perp \sqcup Q_1^\perp \sqcup R^\perp} \qquad \frac{R \qquad S}{R \otimes S} \qquad \frac{S^\perp \quad P_3^\perp \sqcup Q_3^\perp}{S^\perp \sqcup P_3^\perp \sqcup Q_3^\perp}$$

If we consider empires *relative to* the present set of axioms, then by the argument in section (4.4.1) we still obtain conjuncts with the splitting property, namely either $(Q_2^\perp \sqcup P_2^\perp) \otimes (Q_1^\perp \sqcup P_1^\perp \sqcup R^\perp)$, or $R \otimes S$, or $(S^\perp \sqcup Q_3^\perp \sqcup P_3^\perp) \otimes (Q_4^\perp \sqcup P_4^\perp)$.

It is also clear that even in the case of $\mathcal{S}$ the same conjuncts are good candidates, if we intend to split the structure $\mathcal{S}$ by splitting a conjunct $X \otimes Y$. However, we

cannot define what is *deductively relevant* to X (notation: $\mathbf{RLV}(X)$) in terms of the empire of X. A natural idea is the following. First, we consider the set of all formula-occurrences in all *chains* that start at X, but do not contain $X \otimes Y$ (notation: $\mathbf{CH}(X)$); next we add the set of all formulas that are connected to $\mathbf{CH}(X)$ by an axiom-link but do not belong to a chain, e.g., atomic formulas — let $\Phi(X)$ be the resulting set; finally, we let $\mathbf{RLV}(X)$ contain all occurrences of subformulas in $\Phi(X)$. To prove the Sequentialization Theorem for $\mathbf{DL}$ we will show that if the conclusions of a proof-net $\mathcal{S}$ contain no disjunctions, then there is always a conjunction $X \otimes Y$ such that $\mathcal{S} = \mathbf{RLV}(X) \cup \mathbf{RLV}(Y) \cup \{X \otimes Y\}$ is a partition.

It is interesting to remark that if the set of axioms of $\mathcal{S}'$ forms a minimal path (Sequentialization Theorem for $\mathbf{DL}^+$), then the definition of what is *deductively relevant* to X can be restricted as follows. First, we consider the set of all formula-occurrences in all *chains without repetitions* that start at X, but do not contain $X \otimes Y$ (notation: $\mathbf{Ch}(X)$). Next we add the set of all formulas that are connected to $\mathbf{Ch}(X)$ by an axiom-link (notation: $\Phi(X)$). For suitable $X \otimes Y$, this will also produce a partition. This analysis provides an alternative way of obtaining the same splitting as

$$\mathcal{S}' = e(X) \cup e(Y) \cup \{X \otimes Y\}.$$

We need to specify conditions for candidates for a successful splitting: for instance, $P_2 \otimes Q_2$ is a *bad* candidate in $\mathcal{S}$, since the link $\frac{Q_2^\perp \quad P_2^\perp}{Q_2^\perp \sqcup P_2^\perp}$ is 'trapped' by the conjunction $(Q_2^\perp \sqcup P_2^\perp) \otimes (Q_1^\perp \sqcup P_1^\perp \sqcup R^\perp)$. The situation just described will be called a *loop* and $P_2 \otimes Q_2$ is said to be *inside* the loop.

A new graph-theoretic analysis is required to handle loops and other complications arising in the sequentialization process. Consider for instance the proof-structure

$$\mathcal{M} \;=\; \mathcal{L}_1 \;{}_{(Q_1^\perp \sqcup P_1^\perp)}\!\otimes_{(R_2^\perp \sqcup R_1^\perp)}\; \mathcal{M}_1 \;{}_{(S_2^\perp \sqcup S_1^\perp)}\!\otimes_{(Q_2^\perp \sqcup P_2^\perp)}\; \mathcal{L}_2, \tag{‡}$$

where

$$\mathcal{M}_1 :$$

$$\frac{R_2^{\perp} \quad R_1^{\perp}}{R_2^{\perp} \sqcup R_1^{\perp}} \qquad \frac{R_1 \quad S_1}{R_1 \otimes S_1} \qquad \frac{R_2 \quad S_2}{R_2 \otimes S_2} \qquad \frac{S_2^{\perp} \quad S_1^{\perp}}{S_2^{\perp} \sqcup S_1^{\perp}}$$

and

$$\mathcal{L}_1 : \qquad\qquad\qquad\qquad \mathcal{L}_2 :$$

$$\frac{P_1 \quad Q_1}{P_1 \otimes Q_1} \qquad \frac{Q_1^{\perp} \quad P_1^{\perp}}{Q_1^{\perp} \sqcup P_1^{\perp}} \qquad\qquad \frac{Q_2^{\perp} \quad P_2^{\perp}}{Q_2^{\perp} \sqcup P_2^{\perp}} \qquad \frac{P_2 \quad Q_2}{P_2 \otimes Q_2}$$

Consider the conclusions of $\mathcal{M}$:

$$P_1 \otimes Q_1,\ (Q_1^{\perp} \sqcup P_1^{\perp}) \otimes (R_2^{\perp} \sqcup R_1^{\perp}),\ R_1 \otimes S_1,$$
$$R_2 \otimes S_2,\ (S_2^{\perp} \sqcup S_1^{\perp}) \otimes (Q_2^{\perp} \sqcup P_2^{\perp}),\ P_2 \otimes Q_2.$$

Here $(Q_1^{\perp} \sqcup P_1^{\perp}) \otimes (R_2^{\perp} \sqcup R_1^{\perp})$ and $(S_2^{\perp} \sqcup S_1^{\perp}) \otimes (Q_2^{\perp} \sqcup P_2^{\perp})$ have the splitting property, but $R_1 \otimes S_1$ and $R_2 \otimes S_2$ do not. $\mathcal{M}$ is a simple example of a *maze*. We say that $(Q_1^{\perp} \sqcup P_1^{\perp}) \otimes (R_2^{\perp} \sqcup R_1^{\perp})$ and $(S_2^{\perp} \sqcup S_1^{\perp}) \otimes (Q_2^{\perp} \sqcup P_2^{\perp})$ are *outside* the maze and $R_1 \otimes S_1$ and $R_2 \otimes S_2$ are *inside the maze*.

8.1. The Sequent Calculus DL.

It is easy to see that in the calculi **DL** and **DL**$^+$ there are provable disjunctions such that the disjuncts are logically irrelevant to each other. This is never possible in **MLL**, and the property that A is logically relevant to B and B is logically relevant to A may be regarded as characteristic of the meaning of $A \sqcup B$. This view is justified by the role of the connectedness property in proof-nets.

In the context of **DL** and **DL**$^+$ disjunction has an ambiguous meaning. Suppose A is provable and in **DL**$^+$ we derive the disjunction of A and B after introducing B by Weakening. Such a disjunction is more adequately represented by $A \oplus B$ than by $A \sqcup B$. Moreover, if both A and B are provable and in **DL** we derive the

disjunction of A and B after joining the contexts by Mingle, then the meaning of such a disjunction is different from that of both $\oplus$ and $\sqcup$.

Therefore it is convenient to use the language $\mathcal{L}_D$ of Direct Logic (Ketonen and Weyhrauch [1984]) and write $\wedge$ for $\otimes$ and $\vee$ for $\sqcup$.

The sequent calculus **DL** for propositional Direct Logic has the axioms

$$\vdash P, \neg P$$

The logical rules of inference are

$\wedge$ – **rule:**

$$\frac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \wedge B}$$

$\vee$ – **rule:**

$$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B}$$

The structural rules are Exchange and Mingle.

Exchange:

$$\frac{\vdash \Gamma, A, B, \Delta}{\vdash \Gamma, B, A, \Delta}$$

Mingle:

$$\frac{\vdash \Gamma \qquad \vdash \Delta}{\vdash \Gamma, \Delta}$$

Let $\mathbf{DL}^+$ be the system which is like **DL**, but contains the rule of Weakening instead of Mingle.

Weakening

$$\frac{\vdash \Gamma}{\vdash \Gamma, A}$$

(Clearly, Mingle can be eliminated if we have Weakening and **DL** is a subsystem of $\mathbf{DL}^+$.) The rule of Contraction

$$\frac{\vdash \Gamma, A, A}{\vdash \Gamma, A}$$

is not allowed in **DL** nor in $\mathbf{DL}^+$.

8.2. Paths and Structures.

A proof-structure for **DL** or $\mathbf{DL}^+$ is defined as a proof-structure for propositional $\mathbf{MLL}^-$ (section 3), but in the case of $\mathbf{DL}^+$ formula occurrences are allowed that are not conclusions of any link. In more detail, in this context a proof-structure is a nonempty set of *formula-occurrences*, together with a set of *links* between formula-occurrences. Axiom links have the form

$$\overline{P, \neg P}$$

and non-axiom links have one the forms

$$\frac{X \quad Y}{X \wedge Y} \quad \frac{X \quad Y}{X \vee Y}$$

the $\wedge$-links, $\vee$-links, respectively. Formula-occurrences and links satisfy the properties

(1)′ every formula-occurrence the conclusion of at most one link;

(2) every formula-occurrence is the premise of at most one link.

(3) A link $\frac{X \quad Y}{X \circ Y}$ is regarded as *different* from the link $\frac{Y \quad X}{Y \circ X}$. The axiom link $\overline{P, \neg P}$ is identified with the axiom link $\overline{\neg P, P}$.

As before, we define $X \sim Y$ as "X ***and*** Y ***are different occurrences of the same formula***", and $X \prec Y$ as "X ***is an ancestor of*** Y". In this context, $X \prec Y$ is clearly the same as "X ***is a subformula of*** Y". We write $X \prec \Gamma$ for $X \prec Y$, for some Y in Γ.

The notion of ***path*** in a proof-structure is defined in section (5.1) as a nonempty set of axiom links satisfying the ***relevance condition on conjunctions***:

> ***for any conjunction having any ancestor in an axiom, both conjuncts have ancestors in some axioms.***

As before we write $\mathcal{P} \mapsto A$ if there is an axiom $\overline{P, \neg P}$ in $\mathcal{P}$ such that either $P \prec A$ or $\neg P \prec A$.

Conversely, given a set of formula-occurrences Γ, consider the set $\Gamma^{\prec}$ of all occurrences of subformulas in Γ and identify non-axiom links with the relation "$\ldots\ldots$ ***and*** $\_\,.\,\_\,.\,\_$ ***are immediate subformulas of*** $\_\ \_\ \_$ ".

Definition. A path is called *pervasive* if for all atomic subformulas Q in S, $\mathcal{P} \mapsto Q$ (*relevance condition*).

Chains and cycles are defined as in section (5.1).

8.3. The Main Theorem for Direct Logic.

Definition. A proof-structure $\mathcal{S} = (\Gamma^{\prec}, \mathcal{P})$ for **DL** is a *proof-net* if and only if there is no conjunctive cycle.

The main theorem (Theorem 4.4.) of Ketonen and Weyhrauch [1984] can be formulated as follows:

Theorem. *Let Γ be a sequent containing only prenex formulas.*

(i) $\vdash \Gamma$ *is provable in* **DL** *if and only if there is a pervasive path $\mathcal{P}$ for Γ such that $\mathcal{S} = (\Gamma^{\prec}, \mathcal{P})$ is a proof-net.*

(ii) $\vdash \Gamma$ *is provable in* $\mathbf{DL}^{+}$ *if and only if there is a minimal path $\mathcal{P}$ for Γ such that $\mathcal{S} = (\Gamma^{\prec}, \mathcal{P})$ is a proof-net.*

In the rest of this section we will give a proof of the theorem in the propositional case.

8.4. Abstract Chains.

The logical connections within a proof-structure are considered in abstract graph-theoretic terms. *Chains*, *cycles*, *loops* and *mazes* are given a more abstract definition.

Let $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ be a graph. Let $\mathbf{F}$ be a set of subsets of V, with the property that for all $F_1, F_2 \in \mathbf{F}$, $F_1 \cap F_2 \neq \emptyset$ implies $F_1 \subset F_2$ or $F_2 \subset F_1$. Also, let $\mathbf{C}$ be a set of distinguished elements of $\mathbf{F}$, such that each $C \in \mathbf{C}$ is partitioned by some elements $A_1, A_2, \ldots$ of $\mathbf{F} \cup \mathbf{V}$. For the purpose of the sequentialization theorem, we may assume that the partition of each C is binary.

Remark 1. The *intended interpretation* of a $\mathbf{G}$ is as follows.

$\mathbf{V}$ is the set of all occurrences of atomic subformulas in a sequent Γ.

$\mathbf{F}$ is the set of all occurrences of non-atomic subformulas of the formulas in Γ

$\mathbf{C}$ is the set of elements of $\mathbf{F}$ that are *conjunctions*.

$\mathbf{E}$ is a path $\mathcal{P}$ for Γ, as defined in section (5.1).

With an eye to the intended interpretation, we call $A_1, A_2, \ldots$ the *conjuncts* of C.

(i) Given $(\mathbf{V}, \mathbf{E})$ with $\mathbf{F}$ and $\mathbf{C}$ the *relevance condition on conjunctions* is the following property: for each $C \in \mathbf{C}$, if there is an edge E with an vertex in C, then for each conjunct A_i of C there is an edge E_i with a vertex in A_i.

(ii) Let $(\mathbf{V}, \mathbf{E})$ with sets $\mathbf{F}$ and $\mathbf{C}$ satisfy the relevance condition on conjunctions. $\mathbf{E}$ is called *minimal* (relatively to $\mathbf{C}$) if no proper subset of $\mathbf{E}$ satisfies the relevance condition on conjunction. We say that $(\mathbf{V}, \mathbf{E})$ is minimal (relatively to $\mathbf{C}$) if $\mathbf{E}$ is such.

(iii) For $A, B \in \mathbf{F} \cup \mathbf{V}$ with $A \cap B = \emptyset$, say that A and B are *connected* (write $A \parallel B$) if there is an edge e in E with vertices v_1 and v_2 such that $v_1 = A$ or $v_1 \in A$ and $v_2 = B$ or $v_2 \in B$.

(iv) Let $X, Y \in \mathbf{C}$ such that $X \cap Y = \emptyset$ and let $A, B \in \mathbf{F}$ such that $A \subset X$ and $B \subset Y$. We write $X_A -_B Y$ if $A \parallel B$.

(v) We write ${}_AX_B$ if for some $C \in \mathbf{C}$ such that $C \subset X$, A and B are different conjuncts of C. A *chain* is a sequence $X_1, \ldots, X_n$ such that ${}_{A_1}X_{1\,B_1} - \ldots -{}_{A_n} X_{n\,B_n}$.

(vi) A chain $X_1, \ldots, X_n$ is *pure* if for all $i \neq j$ with $1 \leq i, j \leq n$, $X_i \neq X_j$.

(vii) If $\mathcal{C} = X_1, \ldots, X_n$ is a pure chain, then ${}_AY_B - \mathcal{C} -_A Y$ is called a *cycle* of length $n+1$. In the case when $\mathcal{C}$ is empty, this definition implies that ${}_AY_B$ is a cycle of length 1 if $A \parallel B$.

(viii) Let $\mathcal{C}$ be a pure chain. $Y_B - \mathcal{C} -_A Y$ is called a *loop* if not ${}_AY_B$. Y is the *exit* of the loop. When $Y_B - \mathcal{C} -_A Y$ is a loop, we may write $Y_D - \mathcal{C} -_D Y$, where $A \subset D$, $B \subset D$ and D is either Y itself or D is a conjunct of some $C \in \mathbf{C}$.

(ix) An element X in a chain $\mathcal{C}$ is *terminal* if for all Y, Z, if $Y - X - Z$ then we cannot have $Y \not\subset Z$, $Z \not\subset Y$ and $Y, Z \in \mathbf{C}$.

(x) $X \gg^1 Y$ iff there is a loop

$$\mathcal{L} : Z_A - \ldots - X - \ldots -_A Z$$

and either $Y = Z$ or there is a pure chain

$$\mathcal{C} :_A Z_B - \ldots - Y.$$

We summarize this condition by saying that Y ***dominates*** the loop, which X belongs to.

(xi) $X \gg Y$ iff $X \gg^1 Y$ and not $Y \gg^1 X$.

(xii) Let $U_1 - \mathcal{C}_1 - V_1, \ldots, U_n - \mathcal{C}_n - V_n$ be a sequence of pure chains with $W_1 = V_n \cup U_1, W_2 = V_1 \cup U_2, \ldots, W_n = V_{n-1} \cup U_n$ and $W_1, \ldots, W_n \notin \mathbf{C}$. Suppose that if $X \in \mathcal{C}_i \cup \{U_i, V_i\}$ and $Y \in \mathcal{C}_j \cup \{U_j, V_j\}$ for $i \neq j$, then $X \cap Y = \emptyset$. Then

$$W_1 - \mathcal{C}_1 - W_2 - \ldots - W_n - \mathcal{C}_n - W_1$$

is a *maze*. If $C \in \mathcal{C}_i$ for some $i \leq n$, then we say that C is *inside the maze*.

Remark 2. Let $\mathcal{L}$ be a loop with exit Z and let $\mathcal{C}$ be a pure chain starting with Z. *If $\mathcal{C} \cap \mathcal{L} \neq \{Z\}$ then there must be a cycle.*

Indeed, suppose

$$\mathcal{L} : Z_A - \ldots -_C U_D - \ldots -_A Z$$

and

$$\mathcal{C} :_A Z_B - \ldots -_E U - \ldots - Y.$$

Now either

$${}_E U_D - \ldots -_A Z_B - \ldots -_E U \qquad \text{or} \qquad {}_E U_C - \ldots -_A Z_B - \ldots -_E U$$

or both are cycles, depending on whether $E = C$, $E = D$ or $C \neq E \neq D$, respectively. ∎

8.5. Basic Properties of Abstract Chains.

A strict partial ordering of formulas is defined, using the notion of loop. A method is provided that under certain conditions yields a formula lying outside all loops and mazes.

Lemma 1. *Suppose no chain is a cycle. Then $\gg$ is a strict partial ordering.*

Proof. To show that $X \gg Y$ and $Y \gg Z$ implies $X \gg Z$, we need only to show that for all U, V, W, $U \gg V$ and $V \gg^1 W$ implies $U \gg^1 W$. Indeed, assume $X \gg Y$ and $Y \gg Z$. Then certainly $X \gg^1 Z$. Moreover, given $Y \gg Z$, if $Z \gg^1 X$, then $Y \gg^1 X$, a contradiction.

Assume there are loops $\mathcal{L}_1$, $\mathcal{L}_2$ and pure chains $\mathcal{C}_1$, $\mathcal{C}_2$

$$\mathcal{L}_1 : \qquad U_A - X_1 - ... - X - ... - X_n -_A U$$

$$\mathcal{C}_1 : \qquad {}_AU_B - U_1 - U_2 - ... - U_i -_E Y$$

$$\mathcal{L}_2 : \qquad V_C - X_1' - ... -_F Y_G - ... - X_m' -_C V$$

$$\mathcal{C}_2 : \qquad {}_CV_D - V_1 - V_2 - ... - Z$$

Clearly, either ${}_EY_G$ or ${}_EY_F$ or both, depending on whether $E = F$, $E = G$ or $F \neq E \neq G$, say the latter.

Then we claim that

$$U_B - ... -_E Y_G - ... -_C V_D - ... - Z$$

is a pure chain. Since there are no cycles, $\mathcal{C}_2 \cap \mathcal{L}_2 = \{V\}$, by Remark 2. On the other hand, if $U_0 \in \mathcal{C}_1 \cap \mathcal{C}_2$, then either we have a cycle

$${}_HU_{0\,L} - ... - Y - ... - V - ... -_H U_0$$

or

$$V_D - ... -_H U_{0\,L} - ... - U - ... - X$$

is a pure chain and $Y \gg^1 X$, against the hypothesis. ∎

Lemma 2. *Let* $(\mathbf{V}, \mathbf{E})$ *be a graph with sets* $\mathbf{F}$ *and* $\mathbf{C}$ *as above satisfying the relevance condition on conjunctions. If there is no infinite* pure *chain and no cycle, then the there is an element* X_0 *minimal with respect to* $\gg$ *and lying outside all mazes.*

Proof. If $\mathbf{C}$ is empty or no edge has a vertex in some $C \in \mathbf{C}$, then any $F \in \mathbf{F}$ is minimal with respect to $\gg$ and there is no maze. Assume that some edge has a vertex in some $C \in \mathbf{C}$. If X is a terminal element of a chain, then X is minimal with respect to $\gg$ and lies outside all mazes.

Assume that no chain has a terminal element. Since the relevance condition on conjunctions holds, there is an infinite chain $\mathcal{C}$. Since no infinite chain is pure and there is no cycle, $\mathcal{C}$ has the form

$$\ldots - W_D - \mathcal{L}_0 -_D W - \ldots$$

for some nonempty $\mathcal{L}_0$. If $X \in \mathcal{L}_0$ then $X \gg^1 W$. It is immediate to see that if $\mathcal{L}_1$ is another loop

$$\mathcal{L}_1 : \qquad Z_G - \ldots -_E W_F - \ldots -_G Z$$

with $W \in \mathcal{L}_1$ and X dominates $\mathcal{L}_1$, then X and W belong to a cycle

$$_D W_E - \ldots - Z - \ldots - X - \ldots -_D W_E \qquad \text{or} \qquad _D W_F - \ldots - Z - \ldots - X - \ldots -_D W_F.$$

Hence $X \gg W$ and $\gg$ is nonempty. If there was an infinite descending sequence with respect to $\gg$, then by Lemma 1 we would also have an infinite pure chain, a contradiction. Thus $\gg$ is well-founded.

Notice that in this case there are configurations of the form

$$\mathcal{L}_1 -_{D_1} W_1 - \mathcal{C} - W_2\ _{D_2} - \mathcal{L}_2 \qquad (*)$$

where D_1 is the exit of $\mathcal{L}_1$, D_2 the exit of $\mathcal{L}_2$ and both W_1 and W_2 are minimal with respect to $\gg$. Indeed if one of W_1 and W_2, say W_2, is not minimal with respect to $\gg$, then there is a loop $\mathcal{L}'$ and a pure chain connecting W_2 with the exit Z of $\mathcal{L}'$. Z cannot belong to $\mathcal{L}_1$, otherwise there would be a cycle. If Z is in $\mathcal{C}$, then we consider the configuration

$$\mathcal{L}_1 -_{D_1} W_1 - \ldots - Z - \mathcal{L}',$$

where the dots indicate a subchain of $\mathcal{C}$. If Z is not in $\mathcal{C}$, then $\mathcal{C}$ can be extended to a longer pure chain. Since there is no infinite pure chain, we eventually find a loop $\mathcal{L}''$ such that

$$\mathcal{L}_1 -_{D_1} W_1 - \mathcal{C} - \ldots - \mathcal{L}''$$

is also of the form $(*)$. By repeating the argument we eventually find the desired configuration.

To find an element X_0 lying is outside all mazes, we consider a configuration of the form $(*)$ which is maximal, in the sense that we cannot find

$$\mathcal{L}_1' -_{E_1} Z_1 - \mathcal{C}' - Z_2\ _{E_2} - \mathcal{L}_2'$$

such that $\mathcal{C}'$ extends $\mathcal{C}$ and moreover Z_1 and Z_2 are minimal with respect to $\gg$.

We claim that for *if the chain* $(*)$ *has the property that* $\mathcal{C}$ *is maximal* in the sense of the last paragraph, *then one of the formula occurrences* W_1, W_2 *is outside all mazes*. Suppose W_2 is inside a maze: there must be a pure chain $\mathcal{C}^+$ starting from some $X \in \mathcal{L}_2$ which reaches neither $\mathcal{L}_1$ nor $\mathcal{C}$ — in the first case we would have a cycle and in the second W_2 would not be minimal with respect to $\gg$. The assumption of this case is that there is no terminal element, therefore by the argument in the previous paragraph we must have $X - \mathcal{C}^+_{D_0} - \mathcal{L}_0$, where the loop $\mathcal{L}_0$ has exit D_0 minimal with respect to $\gg$. We conclude that $\mathcal{C}$ can be extended and contradict the assumption that the original configuration $(*)$ is maximal. ∎

8.6. Partition Properties.

For the abstract graphs under consideration the notion $\mathbf{RLV}(X)$ is defined. The intended interpretation is the set of formulas *deductively relevant to* the formula X in a proof-structure. Conditions are provided under which a partitioning of X into X_1 and X_2 determines a partition of the entire graph into $\mathbf{RLV}(X_1)$ and $\mathbf{RLV}(X_2)$.

Let the graph $(\mathbf{V}, \mathbf{E})$ and the sets $\mathbf{F}$, $\mathbf{C}$ be as above. For any subset $\mathcal{Y}$ of $\mathbf{F}$, we write $(\mathbf{V} \parallel \mathcal{Y})$ for

$$\{v \in \mathbf{V} : \text{ for all } F \text{ in } \mathbf{F},\ v \notin F, \text{ and for some } Y \text{ in } \mathcal{Y},\ v \parallel Y.\}$$

For any $X \in \mathbf{C}$ we let

$$\mathbf{Ch}(X) = \text{the union of all pure chains } \mathcal{C} - X \text{ in } (\mathbf{V}, \mathbf{E}) \text{ with } \mathbf{F}, \mathbf{C}, \text{ that start with } X;$$

$$\mathbf{CH}(X) = \text{the union of all chains } \mathcal{C} - X \text{ in } (\mathbf{V}, \mathbf{E}) \text{ with } \mathbf{F}, \mathbf{C}, \text{ that start with } X;$$

$$\phi(X) = \mathbf{Ch}(X) \cup (\mathbf{V} \parallel \mathbf{Ch}(X))\};$$

$$\Phi(X) = \mathbf{CH}(X) \cup (\mathbf{V} \parallel \mathbf{CH}(X))\};$$

$$\mathbf{RLV}(X) = \{Y \in \mathbf{F} \cup \mathbf{V} : Y \in Z \text{ or } Y \subset Z \text{ or } Y \parallel Z, \text{for some } Z \in \mathbf{CH}(X)\}.$$

Let $\mathbf{F}'$ and $\mathbf{C}'$ be obtained by removing X_0 from $\mathbf{F}$ and $\mathbf{C}$ and suppose A_1 and A_2 are the only conjuncts of X_0. For $i = 1, 2$ let In other words,

$$\mathbf{Ch}(A_i) = \text{the union of all pure chains } \mathcal{C} - A_i \text{ in } (\mathbf{V}, \mathbf{E}) \text{ with } \mathbf{F}', \mathbf{C}', \text{ that start with } X;$$

$$\mathbf{CH}(A_i) = \text{the union of all chains } \mathcal{C} - A_i \text{ in } (\mathbf{V}, \mathbf{E}) \text{ with } \mathbf{F}', \mathbf{C}', \text{ that start with } X;$$

$$\phi(A_i) = \mathbf{Ch}(A_i) \cup (\mathbf{V} \parallel \mathbf{Ch}(A_i))$$

$$\mathbf{\Phi}(A_i) = \mathbf{CH}(A_i) \cup (\mathbf{V} \parallel \mathbf{CH}(A_i))$$

$$\mathbf{RLV}(A_i) = \{Y \in \mathbf{F}' : Y \subset Z \text{ or } Y \parallel Z, \text{for some } Z \in \mathbf{CH}(A_i)\}$$

We are interested in situations in which given a partition $X_0 = A_1 \cup A_2$,

$$\mathbf{RLV}(X) = \mathbf{RLV}(A_1) \cup \mathbf{RLV}(A_2) \cup \{X\}$$

is a partition. We will need only the case when the maximal sets in $\mathbf{F}$ are all elements of $\mathbf{C}$, in which case it will be enough to show that

$$\mathbf{\Phi}(X) = \mathbf{\Phi}(A_1) \cup \mathbf{\Phi}(A_2) \cup \{X_0\}$$

or that

$$\phi(X) = \phi(A_1) \cup \phi(A_2) \cup \{X_0\}$$

is a partition.

Lemma 3. *Given* $(\mathbf{V}, \mathbf{E})$ *with* $\mathbf{F}$ *and* $\mathbf{C}$ *as before, satisfying the relevance condition on conjunctions, suppose that*

(1) *there is no cycle,*

(2) *the maximal sets in* $\mathbf{F}$ *are all elements of* $\mathbf{C}$,

(3) *no element of* $\mathbf{C}$ *is terminal for any chain,*

(4) $X_0 \in \mathbf{C}$ *is minimal with respect to* $\gg$ *and outside all mazes.*

Let $X_0 = A_1 \cup A_2$ *be a partition. Then*

(i) *if* $\mathbf{E}$ *is minimal relatively to* $\mathbf{C}$, *then for all* $X \in \mathbf{F} \cup \mathbf{V}$, *with* $X \neq X_0$,

$$X \parallel Y, \text{ for some } Y \text{ in } \phi(A_i), \text{ implies } X \in Z \text{ or } X \subset Z, \text{ for some } Z \text{ in } \phi(A_i), \quad (\P)$$

and $\mathbf{RLV}(X_0) = \mathbf{RLV}(A_1) \cup \mathbf{RLV}(A_2) \cup \{X_0\}$ *is a partition.*

(ii) *if no infinite chain is pure, then for all* $X \in \mathbf{F} \cup \mathbf{V}$, *with* $X \neq X_0$,

$$X \parallel Y, \text{ for some } Y \text{ in } \mathbf{\Phi}(A_i), \text{ implies } X \in Z \text{ or } X \subset Z, \text{ for some } Z \text{ in } \mathbf{\Phi}(A_i), \quad (\S)$$

and $\mathbf{RLV}(X_0) = \mathbf{RLV}(A_1) \cup \mathbf{RLV}(A_2) \cup \{X_0\}$ *is a partition.*

Proof. Let $X_0 = A_1 \cup A_2$ be a partition.

(i) To prove (¶), let $X \parallel Y$, with $Y \in \phi(A_i)$.

If $X \in \mathbf{V}$, then by definition $X \in (\mathbf{V} \parallel \mathbf{Ch}(A_i))$.

Suppose $X \in \mathbf{C}$ and not $X = A_i$.

Let Y be A_i. By (3), there is a pure chain $\ldots - X - A_i$.

Let $Y \in \mathbf{Ch}(A_i)$, say $i = 1$, $A_1 - \ldots -_B Y$ and $Y_E - X$. If E is a conjunct of Y different from B, then clearly $A_i - \ldots -_B Y_E - X$ and we are done.

Therefore let $E = B$. We must show that there exists another pure chain $Y_B - X - \ldots - A_1$. Suppose not. Notice that there cannot be a pure chain in $\mathbf{Ch}(A_2)$ of the form $Y_B - X - \ldots - A_2$, since this together with $A_1 - \ldots -_B Y$ implies $X_0 \gg Y$. Therefore $X \notin \mathbf{Ch}(X_0)$.

Consider any $C \in \mathbf{Ch}(X) \cap \mathbf{Ch}(X_0)$. Fore any pair of pure chains $X - \ldots -_{V_1} C$ and $X_0 - \ldots -_{V_2} C$ we cannot have ${}_{V_1}C_{V_2}$, for any V_1, V_2. Let $\mathbf{E}'$ be the result of removing all edges occurring in $\mathbf{Ch}(X)$. It follows that the relevance condition on conjunctions is still satisfied in $(\mathbf{V}, \mathbf{E}')$ with $\mathbf{F}$ and $\mathbf{C}$, contradicting the minimality of $(\mathbf{V}, \mathbf{E})$. (Compare the previous paragraph with the $(\Rightarrow)$ part in the proof of the Multiplicative Disjunction Lemma, section 5.3).

It follows from (¶) that $\mathbf{Ch}(X) = \mathbf{CH}(X)$ and $\mathbf{Ch}(A_i) = \mathbf{CH}(A_i)$, by induction on the length of any $\mathcal{C} \in \mathbf{CH}(X)[\in \mathbf{CH}(A_i)]$. Therefore, to show that $\mathbf{RLV}(X_0) = \mathbf{RLV}(A_1) \cup \mathbf{RLV}(A_2) \cup \{X_0\}$ is a partition, it is enough to show that

$$\phi(X_0) = \phi(A_1) \cup \phi(A_2) \cup \{X_0\}$$

is a partition, by (2) and (¶). Since $X_0 = A_1 \cup A_2$ is a partition, it is immediate that $\phi(X_0) \subset \phi(A_1) \cup \phi(A_2) \cup \{X_0\}$, thus we need to show only that $\phi(A_1) \cup \phi(A_2) = \emptyset$.

Since an element of $(\mathbf{V} \parallel \mathbf{Ch}(X_0))$ cannot have two distinct connections, we need to show only that $\mathbf{Ch}(A_1) \cap \mathbf{Ch}(A_2) = \emptyset$.

Suppose $X_{0\,A_1} - C' -_D Y$ and $Y\,_E - C'' -_{A_2} X_0$ are pure chains. If $_D Y_E$, then there is a cycle, contradicting assumption (1). Otherwise $X_0 \gg Y$, contradicting assumption (4).

(ii) To prove (§), let $X \parallel Y$, $\quad Y \in \mathbf{\Phi}(A_i)$.

If $X \in \mathbf{V}$, then by definition $X \in (\mathbf{V} \parallel \mathbf{CH}(A_i))$.

Suppose $X \in \mathbf{C}$ and not $X = A_i$.

Let Y be A_i. By (3), there is a chain $\ldots - X - A_i$.

Let $Y \in \mathbf{CH}(A_i)$, say $A_i - \ldots -_B Y$ and $Y_E - X$. If E is a conjunct of Y different from B, then clearly $A_i - \ldots -_B Y_E - X$ and we are done.

Therefore let $E = B$. By (3), there is a chain $\mathcal{C}_0$

$$\mathcal{C}_0 = X -_B Y_D - C' \ldots.$$

If $\mathcal{C}_0$ terminates, it terminates in A_i and we are done. Otherwise, there is a first Z such that $X -_B Y_D - \ldots - Z - \ldots - Z$, since there is no infinite pure chain, and $Z - \ldots - Z$ must be a loop, by (1). Therefore

$$X -_B Y_D - C' \ldots -_F Z_G - \ldots -_G Z_F - \ldots C' -_D Y_B - \ldots - A_i.$$

Therefore $X \in \mathbf{CH}(A_i)$.

To verify that $\mathbf{RLV}(X_0) = \mathbf{RLV}(A_1) \cup \mathbf{RLV}(A_2) \cup \{X\}$, is a partition we need to prove only that

$$\mathbf{\Phi}(X_0) = \mathbf{\Phi}(A_1) \cup \mathbf{\Phi}(A_2) \cup \{X_0\}$$

is a partition, by (2) and (§). Since $X_0 = A_1 \cup A_2$ is a partition, it is immediate that $\mathbf{\Phi}(X_0) \subset \mathbf{\Phi}(A_1) \cup \mathbf{\Phi}(A_2) \cup \{X_0\}$, thus we need to show only that $\mathbf{\Phi}(A_1) \cup \mathbf{\Phi}(A_2) = \emptyset$. Since an element of $(\mathbf{V} \parallel \mathbf{CH}(X_0))$ cannot have two distinct connections, we need to show only that $\mathbf{CH}(A_1) \cap \mathbf{CH}(A_2) = \emptyset$.

Suppose not. As shown in part (i), $\mathbf{Ch}(A_1) \cap \mathbf{Ch}(A_2) \neq \emptyset$ contradicts (1) or (4). Therefore we may suppose $X_{0\,A_1} - \mathcal{C} -_{A_2} X_0$ is a (non-pure) chain, say $\mathcal{C}$:

$$X_{0\,A_1} - \mathcal{C}_0 - W_1 - \mathcal{L}_1 - W_1 - \mathcal{C}_1 - W_2 - \ldots - W_{m-1} - \mathcal{C}_{m-1} - W_m - \mathcal{L}_m - W_m - \mathcal{C}_m -_{A_2} X_0$$

where $\mathcal{C}_0, \ldots, \mathcal{C}_m$ are pure chains and we may assume that for all $i \neq j$ with $0 \leq i, j \leq m$, and for all $C' \in \mathcal{C}_i$, $C'' \in \mathcal{C}_j$, we have $C' \not\preceq C''$ and $C'' \not\preceq C'$ (otherwise, we can take a chain shorter than $\mathcal{C}$.) Then by removing $\mathcal{L}_1, \ldots, \mathcal{L}_m$ we obtain a maze containing X_0 and this contradicts assumption (4). ■

8.7. From Proofs to Chains.

Structure Theorem *Let* Γ *be a sequent in* $\mathcal{L}_D$. *Let* $\mathcal{D}$ *be a derivation of* $\vdash \Gamma$ *in* **DL** [*or in* $\mathbf{DL}^+$]. *If* $\mathcal{P}$ *is a list of all* $\overline{P, \neg P}$ *such that* $\vdash P, \neg P$ *or* $\vdash \neg P, P$ *is an axiom of* $\mathcal{D}$, *then* $\mathcal{P}$ *is a pervasive path for* Γ [*or a minimal path for* Γ] *and there is no conjunctive cycle.*

Proof. *Case of* **LD**. We argue by induction on the length of $\mathcal{D}$. We call the set of all $\overline{P, \neg P}$ such that $\vdash P, \neg P$ is an axiom of $\mathcal{D}$ the *path induced by* $\mathcal{D}$. If $\mathcal{D}$ ends with an axiom there is nothing to prove. Let $\mathcal{R}$ be the last inference of $\mathcal{D}$.

If $\mathcal{R}$ is the $\vee$-rule, the path induced by $\mathcal{D}$ is the same as the one induced by the immediate subderivation and the result is immediate from the induction hypothesis.

If $\mathcal{R}$ is a Mingle or the $\wedge$-rule, let $\mathcal{D}_1$ and $\mathcal{D}_2$ be the premises Γ_1 and Γ_2 of $\mathcal{R}$. We may assume that the structures $\mathcal{S}_1 = (\Gamma_1, \mathcal{P}_1)$ and $\mathcal{S}_2 = (\Gamma_2, \mathcal{P}_2)$ share no formula occurrence. The path $\mathcal{P}$ is the union of the paths $\mathcal{P}_1$ and $\mathcal{P}_2$. It is immediate to verify that $\mathcal{P}$ satisfies conditions $(a) - (d)$ in the definition of section (8.2), i.e., that it is a path for Γ, Also $\mathcal{P}$ is pervasive because $\mathcal{P}_1$ and $\mathcal{P}_2$ are.

The fact that there is no conjunctive cycle is immediate from the induction hypothesis in the case of Mingle. In the case of the $\wedge$-rule, to check that there is no conjunctive cycle of length 1 it is enough to note that $\mathcal{P}_1$ and $\mathcal{P}_2$ are disjoint. For cycles of length greater than 1, note that by induction hypothesis a cycle cannot occur only inside $\Gamma_1^{\prec}$ or $\Gamma_2^{\prec}$. Thus we need to consider only conjunctive chains containing $A \wedge B$, say

$$\mathcal{C} :_F X_C - \ldots -_D A \wedge B_E - \ldots -_F X$$

Notice that $A \wedge B$ is the only conjunction such that one conjunct belongs to $\Gamma_1^{\prec}$ and the other to $\Gamma_2^{\prec}$, thus every chain containing conjunctive subformulas of both $\Gamma_1^{\prec}$ and $\Gamma_2^{\prec}$ must contain $A \wedge B$. Let

$$\mathcal{C}_1 : X_C - Y_n - \ldots - Y_1 -_D A \wedge B$$

and

$$\mathcal{C}_2 : A \wedge B_E - Z_1 - \ldots - Z_m -_F X$$

be chains such that $Y_n - \ldots - A \wedge B$ and $A \wedge B - \ldots - Z_m$ are pure (i.e, without repetition). By induction on i and on j we prove that all Y_i belong to $\Gamma_1^{\prec}$ and all Z_j belong to $\Gamma_2^{\prec}$. Now X cannot belong both to $\Gamma_1^{\prec}$ and to $\Gamma_2^{\prec}$, thus we conclude that X is $A \wedge B$ and that $C \prec A$ and $F \prec B$. Furthermore, neither $A_C - Y_n - \ldots - Y_1 -_D A$ nor $B_E - Z_1 - \ldots - Z_m -_F B$ can be a cycle, by induction hypothesis. Finally, $\mathcal{C}$ is not a cycle, since it contains repetition. — Notice that $\mathcal{C}_1$ and $\mathcal{C}_2$ are loops.

We now turn to the case of $\mathbf{DL}^+$. We need a preparatory result. The following tedious proposition is a restatement of a well known result for classical sequent calculus.

Proposition. *Every derivation $\mathcal{D}$ of $\vdash \Gamma$ in $\mathbf{DL}^+$ can be transformed into a derivation $\mathcal{D}'$ of $\vdash \Gamma$ in $\mathbf{DL}^+$ such that for every application $\mathcal{R}_W$ of Weakening either $\mathcal{R}_W$ introduces an active premise of the $\vee$-rule and the other premise has some ancestor in an axiom of $\mathcal{D}'$ or $\mathcal{R}_W$ introduces a formula of Γ and there are no logical rules below $\mathcal{R}_W$. In particular, $\mathcal{D}'$ has the property that every active formula in an application of the $\wedge$-rule has some ancestor in an axiom* (relevance condition on conjunctions).

Proof. By induction on the length of $\mathcal{D}$. Let $\mathcal{R}$ be the last inference of $\mathcal{D}$. By induction hypothesis the result can be applied to immediate subderivation[s] ending with the premise[s] of $\mathcal{R}$; let $\mathcal{D}_1$ [and $\mathcal{D}_2$] be the resulting derivation[s]. Let $\mathcal{R}_1'$ [and $\mathcal{R}_2'$] be the lowermost inference of $\mathcal{D}_1$ [of $\mathcal{D}_2$] different from a Weakening.

Let $\mathcal{R}$ be the $\wedge$-rule with conclusion $A \wedge B$. Suppose A is introduced by an application $\mathcal{R}_W$ of Weakening, and we assume by induction hypothesis that $\mathcal{R}_W$ occurs below $\mathcal{R}_1'$. We replace $A \wedge B$ for A as the conclusion of $\mathcal{R}_W$, we omit $\mathcal{D}_2$ and for every side formula C in the endsequent of $\mathcal{D}_2$, we introduce C by a new Weakening. Suppose neither A nor B are introduced by a Weakening. We obtain a derivation $\mathcal{D}'$ with the required property by permuting $\mathcal{R}$ with all Weakenings occurring below $\mathcal{R}_1'$ and $\mathcal{R}_2'$.

Let $\mathcal{R}$ be the $\vee$-rule with conclusion $A \vee B$ and let $\mathcal{R}_\vee'$, $\mathcal{R}_\vee''$ be the inferences introducing A and B in $\mathcal{D}_1$. If $\mathcal{R}_\vee'$, $\mathcal{R}_\vee''$ are both Weakenings, then we replace $A \vee B$ for A as the conclusion of $\mathcal{R}_\vee'$ and omit $\mathcal{R}_\vee''$. If one of $\mathcal{R}_\vee'$, $\mathcal{R}_\vee''$, say $\mathcal{R}_\vee'$, is a logical

rule and the other is a Weakening, then by induction hypothesis $\mathcal{R}'_\vee$ does not occur below $\mathcal{R}'_1$. We obtain a derivation $\mathcal{D}'$ with the required property by permuting first $\mathcal{R}''_\vee$ and then $\mathcal{R}$ with the other Weakenings occurring below $\mathcal{R}'_1$. If both $\mathcal{R}'_\vee$ and $\mathcal{R}''_\vee$ are logical rules, as before we permute $\mathcal{R}$ with all the Weakenings below $\mathcal{R}'_1$. ∎

Proof of the Theorem, *Case of* $\mathbf{LD}^+$. We assume that the derivation $\mathcal{D}$ satisfies the above proposition. Therefore $\mathcal{P}$ – namely the list of all $\overline{P, \neg P}$ such that $\vdash P, \neg P$ or $\vdash \neg P, P$ is an axiom of $\mathcal{D}$ – satisfies the relevance condition on conjunctions. The other conditions in the definition of path (section 8.2) are easily verified as before, thus $\mathcal{P}$ is a path for Γ.

In addition, we must prove that the path is minimal. Let $\vdash \Gamma$ be $\vdash \Delta, \Pi$, where $\mathcal{P} \not\mapsto \Pi$. By the above proposition, Weakening in $\mathcal{D}$ is used only either to introduce a formula occurring in Π or a subformula B of a disjunction $A \vee B$ or $B \vee A$, where $\mathcal{P} \mapsto A$. We may assume Π is empty, otherwise we consider the subderivation ending with the conclusion of the lowermost logical rule. Now we replace the derivation $\mathcal{D}$ of $\vdash \Delta$ with a derivation $\mathcal{D}'$ of $\vdash \Delta^\ell$ in **MLL** with the *Plus* rule defined as follows. For the subformula occurring in Δ define the translation ℓ by

$P^\ell = P$;

$(\neg P)^\ell = P^\perp$;

$(A \wedge B)^\ell = A^\ell \otimes B^\ell$;

$(A \vee B)^\ell = A^\ell \sqcap B^\ell$, if $\mathcal{P} \mapsto A$ and $\mathcal{P} \mapsto B$;

$(A \vee B)^\ell = A^\ell \oplus B^\ell$ otherwise.

$\mathcal{D}'$ is then inductively obtained from $\mathcal{D}$ by omitting all Weakenings and replacing occurrences of the $\wedge$-rule by the *Times* rule and occurrences of the $\vee$-rule with *Par* or *Plus* rule, as needed. By the Structure Theorem for this fragment (see section 4) there is a proof-structure $\mathcal{S}'(\Delta^\ell)$ satisfying the vicious circle and connectedness conditions. By the proposition in section (5.2) there is no cycle in $\mathcal{S}'$ and by the Multiplicative Disjunction Lemma (section 5.3) from the set of axiom links $\overline{P^\ell, (\neg P)^\ell}$ we obtain a minimal path $\mathcal{P}'$ for $\vdash \Delta$. But $P^\ell = P$ and $(\neg P)^\ell = P^\perp$, thus the axioms of $\mathcal{D}'$ are the same as the axioms of $\mathcal{D}$, after replacing $\neg P$ by $P^\perp$. It is therefore easy to see that $\mathcal{P}$ is a minimal path for Γ and that $(\Gamma^\prec, \mathcal{P})$ contains no cycle. ∎

8.8. From Chains to Proofs.

Sequentialization Theorem (ii) *Let Γ be a sequent in $\mathcal{L}_{\mathcal{L}}$. If there is a pervasive [minimal] path for Γ and there is no conjunctive cycle, then we can construct a derivation $\mathcal{D}$ of $\vdash \Gamma$ in* **DL** *[in* $\mathbf{DL}^+$*].*

Proof.(Case of **LD**.) We have $\Gamma^{\prec}$ and a pervasive $\mathcal{P}$ satisfying the conditions. The proof is by induction on the sum of the logical complexity of all formulas in Γ, plus the number of atoms in Γ which do not occur in $\mathcal{P}$.

Case 0. If $\Gamma = P, \neg P$, then $\mathcal{P}$ must be $\overline{P, \neg P}$ and there is nothing to prove.

If there is a subset $\mathcal{P}_1$ of $\mathcal{P}$ and a subsequent Δ of Γ such that $\mathcal{P}_1$ is a pervasive path for Δ, then it must be the case that $\Gamma = \Delta, \Lambda$, $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ where $\mathcal{P}_2$ is a pervasive path for Λ, since $\mathcal{P}$ is a pervasive path for Γ. In this case the result follows from the induction hypothesis by an application of Mingle.

Case 1. A disjunction occurs in Γ.

If $\Gamma = \Delta, A \vee B$ then the result is immediate from the induction hypothesis applied to Δ, A, B and $\mathcal{P}$, using the $\vee$-rule.

Case 2. No formula in Γ is a disjunction and in some chain there is a terminal conjunction $A_1 \wedge A_2$.

Subcase (a). $\Gamma = \Delta, A_1 \wedge A_2$ and for every $\overline{P, \neg P} \in \mathcal{P}$, if $P \prec A_i$, then also $\neg P \prec A_i$, say $i = 2$. Let $\mathcal{P}_1$ be the restriction of $\mathcal{P}$ to Δ, A_1 and $\mathcal{P}_2$ the restriction of $\mathcal{P}$ to A_2. Clearly $\mathcal{P}_2$ is a pervasive path for A_2, thus $\mathcal{P}_1$ is a pervasive path for Δ, A_1, there are no cycles and the result follows from the induction hypothesis by an application of the $\wedge$-rule.

Subcase (b). $\Gamma = \Delta, A_1 \wedge A_2, \Pi$, where for some i, $\Pi = \{P : P$ is an atom and $P \parallel A_1\}$ and Π and A_i have the following property:

for every X, if $A_i \parallel X$, then $X \in \Pi$.

Say $i = 2$. This is handled as Subcase (a), by letting $\mathcal{P}_1$ be the restriction of $\mathcal{P}$ to Δ, A_1 and $\mathcal{P}_2$ the restriction of $\mathcal{P}$ to A_2, Π.

Case 3. none of the above.

Let $\Gamma = \Sigma, \Pi$, where Σ are all the conjunctions and Π all the atoms and negations of atoms. Let $(\mathbf{V}, \mathbf{E})$ with $\mathbf{F}$ and $\mathbf{C}$ be interpreted as in the Remark 1 in section (8.4).

Notice that, since Γ is finite, there cannot be any infinite chain without repetition. Since by assumption there is no conjunctive cycle, we may apply the Lemma 2 in Section (8.5) to find an element $X_0 = A_1 \wedge A_2$ minimal with respect to $\gg$ and lying outside all mazes. Conditions (1) – (4) of Lemma 3 in section (8.6) are satisfied, thus by part (*ii*) of that lemma there is a partition

$$\mathbf{RLV}(X_0) = \mathbf{RLV}(A_1) \cup \mathbf{RLV}(A_2) \cup \{X_0\}.$$

We claim that $\mathbf{RLV}(X_0) = \Gamma$. Suppose not. Since no formula in Γ is a disjunction, we can find Δ, Λ such that $\Gamma = \Delta, \Lambda$ and $\mathbf{RLV}(X_0) = \Delta$ and $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$, where $\mathcal{P}_1$, $\mathcal{P}_2$ are the restrictions of $\mathcal{P}$ to Δ, Λ, respectively. Thus we are back to Case 0.

Therefore $\mathbf{RLV}(X_0) = \Gamma$. If we let

$$\mathcal{P}_i = \{\overline{P_0, P_1} \in \mathcal{P} : P_j \in \Phi_j \wedge P_{1-j} \in \mathbf{RLV}(A_i), \text{ for } j = 0, 1\}$$

it is easy to show that $\mathcal{P}_i$ is a pervasive path for Φ_i, for $i = 1, 2$. In particular, the relevance condition on conjunctions is satisfied since each set $\mathbf{RLV}(A_i)$ is closed under $\|$.

The proof for $\mathbf{DL}^+$ is similar. Since $\mathcal{P}$ is minimal, it cannot happen that $\Gamma = \Delta, \Lambda$, $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$, $\mathcal{P}_1$ is a path for Δ and $\mathcal{P}_2$ is a path for Λ. Instead, it may happen that $\Gamma = \Delta, \Lambda$ and $\mathcal{P} \not\leftrightarrow \Lambda$. This case is easily handled using Weakening. Finally, in case 3 we use part (*i*) of the Lemma 3 in section (8.6). ∎

Remark. By the translation ℓ, the sequentialization theorem is also a consequence of the sequentialization theorem for propositional **MLL** plus the *Plus* rule. The proof above is giveen in order to make a comparison between the representation of proofs by proof structures and empires and the representation by chains and cycles.

9. Conclusion.

We have modified Girard's definition of Proof-Structures for Linear Logic without exponentials (section 3), and provided a new characterization of Proof-Nets (section 3.5) based on the notion of *empire* of a formula-occurrence — the set of formula-occurrences deductively relevant to the formula- occurrence in question. The notion of empire (section 3.4) has been inductively defined as primitive for

both the multiplicative and the additive fragment. Searches through an empire are formalized as trees (section 4.1), instances of a well-known notion of inductive search. The new conditions have been proved necessary and sufficient for provability in Sequent Calculus for Linear Logic without exponentials (section 4).

A second equivalent characterization of Proof-Nets for the Multiplicative fragment has been obtained by correcting and modifying a condition first considered by Ketonen and Weyhrauch for Direct Logic, in which a set of possible axioms is regarded a *path* though the conclusion and *chains* of conjunctive formulas are tested to guarantee deductive adequacy (section 5).

Another representation of derivability in the Multiplicative and Additive fragment is provided, through the notion of *decomposition* of Proof-Structures into Families of Quasi-Structures. In such an approach, the multiplicative links uniquely determine the deductive structure within each Quasi-Structure, while additive links determine the connections between different elements of a Family (section 6).

Cut-Elimination is studied using Families of Quasi-Structures as representation of derivations. The Strong Normalization Theorem yields here uniqueness of the normal form, in the sense that normal derivations obtained trhough different sequences of reductions are mapped to the same set of quasi-structures (section 7.3).

We have obtained a generalization of the previous results to Direct Logic with Mingle. Abstract graph-theoretic properties of Ketonen and Weyhrauch's representation are exploited to prove it adequate and correct (section 8).

PART II

10. Proof Theoretic Tools.

In this section we review some classical proof-theoretic results, *Herbrand's Theorem* and Kreisel's *No Counterexample Interpretation* (NCI). The proofs are well-known. Next we consider Herbrand's theorem in the context of Linear and Direct Logic and a restriction of the NCI to a fragment of Peano Arithmetic.

Definitions. Let $\mathcal{L}$ be a first order language.

(i) Let us say that a formula A is in Π_1 *form* [Π_2 *form*] if it is of the form $\bigwedge x_1 \ldots . \bigwedge x_n.B$, $[\bigwedge x_1 \ldots \bigwedge x_n. \bigvee y_1 \ldots \bigvee y_m.B]$ where B is quantifier free and $n, m \geq 0$.

(ii) A quantifier and its bound variable are *essentially universal* if they are universal in a positive context or existential in a negative context; otherwise, they are *essentially existential.*

Let Γ be a sequence of closed formulas of $\mathcal{L}$, and let $Qx_1, \ldots, Qx_n$ are all the distinct essentially existential quantifiers and $Qy_1, \ldots, Qy_m$ are all the distinct essentially universal quantifiers of Γ. We say that $x_1, \ldots, x_n$ are all the distinct essentially existential variables and $y_1, \ldots, y_m$ all the distinct essentially universal variables of Γ. (Of course, each variable in each list may have several occurrences in Γ.) For $i \leq m$, let $Qy_i.B_i$ be the scope of the quantifier Qy_i and let $Qx_{i,1}, \ldots, Qx_{i,k_i}$ be all the existential quantifiers containing $Qy_i.B_i$ in their scope.

(iii) The expression $\mathbf{y}_i[x_{i,1}, \ldots, x_{i,k_i}]$ is called the *Herbrand's function* associated with y_i.

Let $\mathcal{L}'$ be the language $\mathcal{L}$ extended with the Herbrand functions for Γ. We will use the boldface letter "$\mathbf{y}$" for the Herbrand function as a formal symbol of the extended language $\mathcal{L}'$, where $\mathbf{y}$ is associated with an essentially universal quantifier $\forall y$.

(iv) The *Herbrand's form* Γ_H of Γ is the result of replacing all essentially universal variables of Γ with their Herbrand function and of deleting the corresponding quantifiers. If $\vec{\mathbf{t}} = \mathbf{t}_1, \ldots, \mathbf{t}_n$ is a list of terms then $\Gamma_H^{\vec{\mathbf{t}}}$ is the result of replacing $\vec{\mathbf{t}}$ for the essentially existential variables $\vec{x} = x_1, \ldots, x_n$ of Γ_H and of deleting the corresponding quantifiers.

Paty (ii) of the following proposition, together with the Cut-Elimination Theorem for Predicate Calculus, yields *Herbrand's Theorem.*

Proposition. (i) *Let* Γ *be a set of* Π_1^0 *sentences and let* ψ *be a quantifier-free formula of* $\mathcal{L}$. *If* $\Gamma \vdash \forall x.\exists y.\psi(x,y)$ *is provable in* **LK** *without Cut, and* a *is a parameter not used in the derivation, then there are terms* $t_1, \ldots, t_n \in \mathcal{L}$ *such that* $\Gamma \vdash \psi(a,t_1), \ldots, \psi(a,t_n)$ *is provable in* **LK** *without Cut.*

(ii) *Let* $\vdash \Gamma$ *be any sequent in* $\mathcal{L}$ *and let* $\mathcal{L}'$ *be an extension of* $\mathcal{L}$ *containing Herbrand functions for the essentially universal quantifiers of* Γ. *In* $\mathcal{L}'$ *there are terms* $\vec{\mathbf{t}}$ *and a quantifier-free sequent* $\mathcal{E}_{\vec{\mathbf{t}}}(\Gamma)$ (the Herbrand expansion of Γ) *such that* $\vdash \Gamma$ *is derivable in* **LK** *without Cut if and only if* $\vdash \mathcal{E}_{\vec{\mathbf{t}}}(\Gamma)$ *is.* ∎

Remark 1. Herbrand [1930] works with a language $\mathcal{L}$ with connectives $\neg$, $\vee$, $\wedge$ and quantifiers $\forall$, $\exists$. Given a formula A, let $\mathcal{F}$ be a list of all constant and functions occurring in A (if no constant occurs in A, we take a new one) together with the Herbrand functions associated with essentially universal quantifiers of A. Define $\mathcal{D}_p$, the *domain of order* p, inductively as follows. $\mathcal{D}_1$ is the set of all constants in $\mathcal{F}$. $\mathcal{D}_{n+1}$ is the union of $\mathcal{D}_n$ and of the set of all terms obtained by applying some function in $\mathcal{F}$ to the elements of $\mathcal{D}_n$. Let $\mathcal{D}_p$ be $\{\mathbf{t}_1,\ldots,\mathbf{t}_q\}$. $\mathcal{E}_{\mathbf{t}_1,\ldots,\mathbf{t}_q}(A)$, the *expansion of* A *of order* p, results from A by the following transformations:

(1) replace all the essentially universal variables in A with the associated Herbrand function and delete the corresponding quantifiers. Let A_H be the resulting formula;

(2) by induction on the complexity of A_H, replace every subformula of the form $\exists x.C$ in positive occurrence with $C(\mathbf{t}_1) \vee \ldots \vee C(\mathbf{t}_q)$ and every subformula of the form $\forall x.C$ in negative occurrence with $C(\mathbf{t}_1) \wedge \ldots \wedge C(\mathbf{t}_q)$.

Remark 2. Herbrand's theorem can be proved as a corollary of Kleene's Permutability Theorem [1952], which allows the permutation of any two propositional inferences, provided that the principal formulas of one inference is not an ancestor of the principal formula of the other.

In the proof of the "*only if*" part, given a cut-free derivation $\mathcal{D}$ of $\vdash \Gamma$, let p be the number of applications of the $\exists$-right and $\forall$-left rules in $\mathcal{D}$. We need to obtain a cut free derivation $\mathcal{D}'$ of $\vdash \mathcal{E}_{\vec{\mathbf{t}}}(\Gamma)$, where $\mathcal{E}_{\vec{\mathbf{t}}}(\Gamma)$ is the expansion of Γ of order p. $\mathcal{D}'$ is easily obtained from $\mathcal{D}$ by the following transformations.

(1) Each eigenvariable is replaced by a suitable term of $\mathcal{L}'$ and every inference introducing an essentially universal formula is omitted.

(2) An application of the $\exists$-right rule is replaced in $\mathcal{D}'$ by a sequence of applications of the $\vee$-right rule and an application of the $\forall$-left rule is replaced by a sequence of applications of the $\wedge$-left rule.

The proof of the " *if* " part can be described as follows (Kleene [1968] gives a proof in the case of sequents of prenex formulas). Let $\mathcal{D}$ be a cut-free derivation of $\vdash \mathcal{E}_{\bar{\mathbf{t}}}(\Gamma)$ and consider the list of terms $\mathbf{t}_1, \ldots, \mathbf{t}_m$ of the form

$$\mathbf{y}_1[\mathbf{t}_{1,1}, \ldots, \mathbf{t}_{1,k_1}], \ldots, \mathbf{y}_m[\mathbf{t}_{m,1}, \ldots, \mathbf{t}_{n,k_m}]$$

such that for each i, $\mathbf{y}_i$ corresponds to an essentially universal quantifier $Qx_i.A_i$ occurring in Γ. We need to find a derivation $\mathcal{D}'$ of $\vdash \mathcal{E}_{\bar{\mathbf{t}}}(\Gamma)$ such that a derivation $\mathcal{D}''$ of $\vdash \Gamma$ can be obtained from $\mathcal{D}'$ by inserting suitable applications of the rules for $\exists$ and $\forall$ and of Contraction. In particular, $\mathcal{D}'$ must be such that if for each i, $\mathbf{y}_i[\mathbf{t}_{i,1}, \ldots, \mathbf{t}_{i,m_i}]$ is regarded as an eigenvariable a_i, then $\mathcal{D}''$ satisfies the restrictions on the eigenvariables.

To construct $\mathcal{D}'$ we assign a strict partial ordering to the terms $\mathbf{t}_1, \ldots, \mathbf{t}_m$, by letting $\mathbf{t}_j$ precede $\mathbf{t}_i$ if and only if $\mathbf{t}_i$ *is a subterm of* $\mathbf{t}_j$. Next we consider the formula-occurrences in $\mathcal{D}$ that in $\mathcal{D}''$ become active in some application of the rules for $\exists$ and $\forall$. We extend the ordering of the terms $\mathbf{t}_i$ to an ordering of such formula-occurrences. The ordering of formula-occurrences has the additional property that if $A(\mathbf{t})$, $B(\mathbf{t})$ become active in inferences whose principal formula are essentially universal, essentially existential, respectively, then $B(\mathbf{t})$ precedes $A(\mathbf{t})$.

Finally, we show that Kleene's Permutability Theorem can be applied to $\mathcal{D}$ so that if $B(\mathbf{t})$ precedes $A(\mathbf{t})$, then in $\mathcal{D}'$ the lowermost occurrence of $B(\mathbf{t})$ is above the lowermost occurrence of $A(\mathbf{t})$.

Remark 3. Consider the classical equivalences

$$\begin{array}{cc} \neg\forall x.B \equiv \exists x.\neg B & \neg\exists x.B \equiv \forall x.\neg B \\ (Qx.B) \vee C \equiv Qx.(B \vee C) & C \vee (Qx.B) \equiv Qx.(C \vee B) \\ (Qx.B) \wedge C \equiv Qx.(B \wedge C) & C \wedge (Qx.B) \equiv Qx.(C \wedge B), \end{array}$$

where C does not contain x free. Given a formula A, by replacing equivalent subformulas according to the above rules, we may obtain (1) formulas A' in prenex form or (2) a formula A'' in which the quantifiers have minimal scope. Herbrand [1930] defines two kinds of expansions of A, obtained by applying certain transformations to either A' or to A'', respectively.

In Case (1) the expansion of order p has a simple form. Let $\vec{\mathbf{t}} = \mathbf{t}_1, \ldots, \mathbf{t}_q$ be all the terms in $\mathcal{D}_p$ and let $x_i, \ldots, x_n$ be all the essentially existential variables in A. Then $\mathcal{E}_{\vec{\mathbf{t}}}(A')$ is $A'^{\sigma_1(\vec{\mathbf{t}})}_H \vee \ldots \vee A'^{\sigma_r(\vec{\mathbf{t}})}_H$ where $\sigma_1, \ldots, \sigma_r$ are all the n^q assignments of terms in $\vec{\mathbf{t}}$ for the essentially existential variables in A.

Case (2) is interesting, because here any Herbrand function $\mathbf{y}[x_1, \ldots, x_m]$ has minimal number of arguments. The well-known errors in Herbrand's thesis (Dreben, Andrews and Aanderaa [1963]) are related to the claim that the order p of the expansion satisfying Herbrand's Theorem for a formula A is preserved under transformation of A by replacement of subformulas according to the above equivalences. The difficulty of the corrections by Dreben and Denton [1966] (see also Note E, p.571, in van Heijenoort [1967]) illustrates how massive transformations in the structure of proofs are induced by transformation of formulas into prenex form.

Remark 4. If conditional terms `if...then...else...` can be expressed in $\mathcal{L}$, then in part (*i*) of the Proposition above we can take $n = 1$ and in part (*ii*) there are terms $\vec{\mathbf{t}} = \mathbf{t}_1, \ldots, \mathbf{t}_k \in \mathcal{L}'$ (*functionals of the predicate calculus*) such that $\mathcal{E}_{\vec{\mathbf{t}}}(\Gamma)$ is $\Gamma^{\vec{\mathbf{t}}}_H$.

10.1. Permutability of Inferences in Linear and Direct Logic.

Counterexamples. (1) The following is a derivation in $\mathbf{MLL}^-$ in which the applications of the $\otimes$-rule and of the $\sqcup$-rule cannot be permuted.

$$\dfrac{\dfrac{\dfrac{\vdash P^\perp, P \qquad \vdash Q, Q^\perp}{\vdash P^\perp, P \otimes Q, Q^\perp}}{\vdash Q^\perp, P^\perp, P \otimes Q}}{\vdash Q^\perp \sqcup P^\perp, P \otimes Q}$$

(2) The following is a derivation in **MALL** in which the applications of the $\otimes$-rule and of the $\sqcap$-rule cannot be permuted.

$$\dfrac{\dfrac{\vdash P^{\perp}, P}{\vdash P^{\perp} \oplus Q^{\perp}, P} \qquad \dfrac{\vdash Q^{\perp}, Q}{\vdash P^{\perp} \oplus Q^{\perp}, Q}}{\vdash P^{\perp} \oplus Q^{\perp}, P \sqcap Q}$$

A full characterization of permutability of inference in **MLL** is obtained using the mapping of derivations into proof-nets (section 3.7) and the notion of empire.

Fact. *Let $\mathcal{D}$ be a derivation in propositional* MLL *and let $\mathcal{R}_1$ and $\mathcal{R}_2$ be inferences such that*

(1) *$\mathcal{R}_1$ is a $\otimes$-rule with upper sequents Γ_1 and Γ_2;*

(2) *$\mathcal{R}_2$ a $\sqcup$-rule occuring immediately below $\mathcal{R}_1$;*

(3) *both active formulas of $\mathcal{R}_2$ have ancestors only in Γ_1 or only in Γ_2.*

Then $\mathcal{R}_1$ can be permuted with $\mathcal{R}_2$. ∎

Let us say that an inference $\mathcal{R}$ in a deduction *introduces* a formula-occurrence X if X is the principal formula of $\mathcal{R}$.

Theorem. *Let $\mathcal{D}$ be any derivation of $\vdash \Gamma$ in propositional* **MLL**, *and let $\pi(\mathcal{D})$ be the corresponding proof-net. Let $\mathcal{R}$ be an inference in $\mathcal{D}$ introducing $A \circ B$. By permuting the inferences of $\mathcal{D}$ we obtain a derivation $\mathcal{D}'$ of $\vdash \Gamma$, with the property that for all C, if $\pi(D) \in e(\pi(A))$, then the inference introducing C does not occur below $\mathcal{R}$.*

Proof. Let k be the number of formula occurrences C such that $\pi(C) \in e(\pi(A))$ and C is introduced below $\mathcal{R}$ — the inference in which A is "captured" by $A \circ B$. By induction on $e(\pi(A))$ (see section 3.4), we show that we can permute the inferences of $\mathcal{D}$ so that we reduce k. The cases of clauses $(i) - (iii)$ in the definition of empire are trivial.

Clause (iv): $\pi(X) \in e(\pi(A))$ and $X \neq A$ implies $\pi(X \otimes Y) \in e(\pi(A))$. By induction hypothesis we may assume that X is introduced above $\mathcal{R}$. If $\mathcal{R}'$ introduces $X \otimes Y$ and occurs below $\mathcal{R}$, then X is a passive formula of every sequent between $\mathcal{R}$ and $\mathcal{R}'$. If we permute $\mathcal{R}'$ with the inference $\mathcal{R}''$ immediately above it, we do not increase the number of formulas in $e(A)$ that are introduced below $\mathcal{R}$, while the total number

of inferences in the derivation remains the same. After a finite number of steps, the inference introducing $X \otimes Y$ is permuted above $\mathcal{R}$ and we have reduced k.

Clause (v): $\pi(X) \in e(\pi(A)), \pi(Y) \in e(\pi(A))$ and $X \neq A \neq Y$ imply $\pi(X \sqcup Y) \in e(\pi(A))$. By induction hypothesis we assume that both X and Y are introduced above $\mathcal{R}$. It follows that for each application $\mathcal{R}^{\otimes}$ of the $\otimes$-rule between $\mathcal{R}$ and $\mathcal{R}'$ both X and Y have ancestors only in one of the upper sequents of $\mathcal{R}^{\otimes}$. If the inference $\mathcal{R}''$ immediately above $\mathcal{R}'$ is another $\sqcup$-rule, then clearly $\mathcal{R}'$ can be permuted with $\mathcal{R}''$. If $\mathcal{R}''$ is a $\otimes$-rule, then by the previous Fact $\mathcal{R}'$ can be permuted with $\mathcal{R}''$. ∎

The above result can be extended to **MALL** and **LD**$^+$. However, it is convenient to use proof-nets directly, rather than the Sequent Calculus and the Permutability Theorem, in the proof of Herbrand's theorem for **MLL** below.

10.2. Herbrand's Theorem in Linear and Direct Logic.

Counterexamples. Suppose x does not occur in Q.

(1) The formula $(\bigwedge x.P \otimes Q) \multimap (\bigwedge x.P) \otimes Q$ is not derivable in first order Linear Logic, but its Herbrand expansion $P(\mathbf{x}) \otimes Q \multimap P(\mathbf{x}) \otimes Q$ is derivable in propositional **MLL**$^-$.

(2) The formula $(\bigvee x.P) \sqcap Q \multimap (\bigvee x.P \sqcap Q)$ is not derivable in first order Linear Logic, but its Herbrand expansion $P(\mathbf{x}) \sqcap Q \multimap P(\mathbf{x}) \sqcap Q$ is derivable in propositional **MALL**.

For the fragment **MLL** — and for the extension **LD**$^+$ — Herbrand's Theorem holds for prenex formulas. Working in the sequent calculus, we consider sequents $\vdash \Gamma$ where all formulas are prenex. If Γ contains free variables, then the free variables are considered as universally quantified, and the Herbrand function of a free variable z is $\mathbf{z}$.

Herbrand's Theorem for MLL. *Let Γ be a sequence of prenex formulas of* MLL *and let $\mathcal{L}'$ be an extension of $\mathcal{L}$ containing Herbrand functions for the universally quantified variables of Γ. There are terms $\vec{\mathbf{t}} = \mathbf{t}_1, \ldots, \mathbf{t}_n \in \mathcal{L}'$ such that $\vdash \Gamma$ is derivable in first order* MLL *without Cut if and only if $\vdash \Gamma_H^{\vec{\mathbf{t}}}$ is derivable in propositional* MLL *without Cut.*

Proof. Consider a proof-net $\mathcal{S}$ with conclusions $\Gamma = A_1, \ldots, A_n$ and without cuts. Consider the proof-net $\mathcal{S}'$ obtained from $\mathcal{S}$ as follows. Let $\bigwedge y_1.B_1, \ldots, \bigwedge y_m.B_m$ be a list of all the conclusions of $\bigwedge$-links in $\mathcal{S}$, with associated eigenvariables

$$b_1, \ldots, b_m.$$

Since the ordering $>_t$ is strict, we may assume that $b_1, \ldots, b_m$ are ordered in such a way that if $b_i >_t b_j$ then $i < j$. Let

$$(*) \qquad \mathbf{y}_1[x_{1,1}, \ldots, x_{1,k_1}], \ldots, \mathbf{y}_m[x_{m,1}, \ldots, x_{m,k_m}]$$

be a list of the Herbrand's functions associated with $\bigwedge y_1, \ldots, \bigwedge y_m$. Each $x_{i,j}$ is an essentially existential variable and corresponds to one and only one link

$$\frac{C(t_{i,j})}{\bigvee x_{i,j}.C.}$$

such that $\bigwedge y_i.B_i \prec C(t_{i,j})$. Consider the pair of lists

$$[0] \qquad \mathbf{y}_1[t_{1,1}, \ldots, t_{1,k_1}], \ldots, \mathbf{y}_m[t_{m,1}, \ldots, t_{m,k_m}]; \qquad t_1, \ldots, t_n.$$

The first list is the result of replacing each $x_{i,j}$ with $t_{i,j}$ in $(*)$ and will be called the list of ***universal terms***. The list of ***universal terms*** has the following property.

Fact 1. *If the eigenvariable b_i occurs in a term $\mathbf{y}_j[t_{j,1}, \ldots, t_{j,k_j}]$, then $b_i >_t b_j$.*

The proof is by an easy induction on the complexity of $\mathbf{y}_j[t_{j,1}, \ldots, t_{j,k_j}]$. It follows from Fact 1 that b_i occurs only in $\mathbf{y}_j[t_{j,1}, \ldots, t_{j,k_j}]$ with $i < j$.

The second list in [0] contains all terms t such that

$$\frac{C(t)}{\bigvee x.C.}$$

is a link of $\mathcal{S}$ and will be called the list of *existential terms*.

We proceed in steps. We start by replacing $\mathbf{y}_1[t_{1,1}, \ldots, t_{1,k_1}]$ for the eigenvariable b_1 throughout $\mathcal{S}$ and the lists [0]. We obtain a proof-net $\mathcal{S}^{(1)}$ and a pair [1] of lists.

At the $(i+1)$-th step we replace the $(i+1)$-th universal term in $[i]$ for b_{i+1} in $\mathcal{S}^{(i)}$ and in the lists $[i]$, obtaining a proof-net $\mathcal{S}^{(i+1)}$ and a pair of lists $[i+1]$. Let $(\star)$ be

$$[m] \qquad \mathbf{y}_1[\mathbf{t}_{1,1},\ldots,\mathbf{t}_{1,k_1}],\ldots,\mathbf{y}_m[\mathbf{t}_{m,1},\ldots,\mathbf{t}_{m,k_m}]; \qquad \mathbf{t}_1,\ldots,\mathbf{t}_n.$$

Using Fact 1 we can show that

Fact 2. *Let $b'_1,\ldots,b'_m$ be a permutation of $b_1,\ldots,b_m$, such that $b'_i >_t b'_j$ implies $i < j$. The terms occurring in the lists $(\star)$ do not change if the universal terms of $[0]$ are ordered according to $b'_1,\ldots,b'_m$.*

Finally, we delete all $\bigwedge$- and $\bigvee$-links in $\mathcal{S}^{(m)}$ and let $\mathcal{S}'$ be the resulting proof-structure. Since the $\bigwedge$- and $\bigvee$-links are unary, $\mathcal{S}'$ still satisfies the vicious circle and connectedness conditions, i.e., $\mathcal{S}'$ is a proof-net. Thus the list of existential terms in $(\star)$ yields the required terms $\vec{\mathbf{t}} = \mathbf{t}_1,\ldots,\mathbf{t}_n$.

Conversely, suppose $\Gamma = A_1,\ldots,A_n$ consists of prenex formulas only and $\mathcal{S}'$ is a proof-net without cuts and with conclusion $\Gamma^{\vec{\mathbf{t}}}_H = A^*_1,\ldots,A^*_n$. We expand $\mathcal{S}'$ to a proof-structure $\mathcal{S}$ with conclusion Γ, by introducing suitable $\bigwedge$- and $\bigvee$-links. Since these links are unary, the vicious circle and connectedness conditions are still satisfied by $\mathcal{S}'$ and we need to check only the parameters condition. Here the eigenvariables are Herbrand's terms of the form $\mathbf{y}[\mathbf{t}_1,\ldots,\mathbf{t}_k]$.

Fact 3. *Let $\mathbf{t}_1$, $\mathbf{t}_2,\ldots,\mathbf{t}_n$ be all the eigenvariables in $\mathcal{S}$, and for all i, let $\mathbf{t}_i$ be associated with $\bigwedge x_i.B_i$. Suppose that $\mathbf{t}_1 >^0 \mathbf{t}_2 >^0 \ldots >^0 \mathbf{t}_n$ and $B_1(\mathbf{t}_1)\Diamond B_2(\mathbf{t}_2)$. Then for all $i \le n$, $\mathbf{t}_1$ occurs in $\mathbf{t}_i$.*

To prove Fact 3, let $\mathbf{t} = \mathbf{y}[\mathbf{t}_1,\ldots,\mathbf{t}_k]$ and $\mathbf{s} = \mathbf{v}[\mathbf{s}_1,\ldots,\mathbf{s}_\ell]$ be the eigenvariables associated with $\bigwedge y.B$, $\bigwedge v.C$, respectively. If $\mathbf{t} <^0 \mathbf{s}$, then either $B(\mathbf{t})\sqsubset C(\mathbf{s})$ or $B(\mathbf{t})\Diamond C(\mathbf{s})$. In addition, since Γ consists of prenex formulas, $\mathbf{s}$ occurs in $\bigwedge y.B$ and, moreover, we have the following properties.

(1) If $B(\mathbf{t})\sqsubset C(\mathbf{s})$, then $B(\mathbf{t}) \prec C(\mathbf{s})$, and every proper subterm $\mathbf{s}_i$ of $\mathbf{s}$ is also a proper subterm of $\mathbf{t}$.

(2) If $B(\mathbf{t})\Diamond C(\mathbf{s})$, then $\mathbf{s}$ is a proper subterm of $\mathbf{t}$ — in fact, since no Herband's term occurs in the conclusions of S, there is a link

$$\frac{D(\mathbf{s})}{\bigvee z.D}$$

with $\bigwedge y.B \prec D(\mathbf{s})$ and thus in $\mathbf{y}[\mathbf{t}_1,\ldots,\mathbf{t}_k]$ some $\mathbf{t}_i$ is $\mathbf{s}$.

By assumption and (2), $\mathbf{t}_1$ is a proper subterm of $\mathbf{t}_2$. For $j \geq 2$, if $\mathbf{t}_2,\ldots,\mathbf{t}_j$ are such that $B_j(\mathbf{t}_j) \sqsubset B_{j-1}(\mathbf{t}_{j-1}) \sqsubset \ldots \sqsubset B_2(\mathbf{t}_2)$ then $\mathbf{t}_1$ is a proper subterm of $\mathbf{t}_2,\ldots,\mathbf{t}_j$, by (1). Also, if $B_{j+1}(\mathbf{t}_{j+1}) \Diamond B_j(\mathbf{t}_j)$, then $\mathbf{t}_1$ is a proper subterm of $\mathbf{t}_{j+1}$, since $\mathbf{t}_j$ is a proper subterm of $\mathbf{t}_{j+1}$ by (2). Proceeding in this way, we conclude that $\mathbf{t}_1$ is a proper subterm of $\mathbf{t}_n$.

To finish the proof of the theorem, suppose now the ordering $<_t$ is not strict in $\mathcal{S}$, i.e., $\mathbf{t} = \mathbf{t}_1 >^0 \mathbf{t}_2 >^0 \ldots >^0 \mathbf{t}_n = \mathbf{t}$, where $\mathbf{t}_i$ is the eigenvariable associated with $\bigwedge y.B_i$. Since $X \sqsubset Y$ implies $e(X) \subsetneq e(Y)$, we must have $B_i(\mathbf{t}_i) \Diamond B_{i+1}(\mathbf{t}_{i+1})$, for some i and we may assume $i = 1$. It follows that $\mathbf{t}_1$ is a proper subterm of $\mathbf{t}_n$, a contradiction. ∎

Remark 1. If Γ contains non-prenex formulas, $\mathcal{S}$ is a proof-structure with conclusion Γ and $\bigwedge x.A$ and $\bigwedge y.B$ are two formula-occurrences in $\mathcal{S}$ with eigenvariables a and b, respectively, then it may happen that b occurs inside and outside $e(A(a))$, $\bigwedge x.A \not\prec \bigwedge y.B$ and b occurs in a side door of $e(A(a))$ but not in $\bigwedge x.A$. In this case, the information that $a <^0 b$ cannot be reconstructed from $\Gamma_H^{\bar{\mathbf{t}}}$ and must be directly obtained by the analysis of the empires.

Remark 2. Can we extend the Herbrand Theorem above to **MALL**? How should the Herbrand expansion be defined in this case? Consider the proof-structure $\mathcal{S}$ with conclusions $\Gamma = C_1,\ldots,C_n, Qx_1 \ldots Qx_n.A \sqcap B$, obtained from

$$
\begin{array}{ccc}
\mathcal{S}_1 & & \mathcal{S}_2 \\[4pt]
C_1,\ldots,C_n & \dfrac{\dfrac{\dfrac{A \qquad\qquad B}{A \sqcap B}}{\cdots}}{Qx_1 \ldots Qx_n.A \sqcap B} & C_1,\ldots,C_n
\end{array}
$$

by introducing appropriate Contraction links and suppose $C_1,\ldots,C_n$ and $Qx_1 \ldots Qx_n.A \sqcap B$ are in prenex form. Clearly if we suppose $\mathcal{S}_1$ and $\mathcal{S}_2$ do not contain any occurrence of a $\sqcap$-link and we apply the procedure in the *only if* part of the Theorem to the displayed structure, we obtain a structure with conclusions

$$C_1^*,\ldots,C_n^*,(A \sqcap B)^*,C_1^{**},\ldots,C_n^{**}.$$

The only reasonable solution is the introduction of $\oplus$- and Contraction links and the construction of a proof-structure with conclusions $\mathcal{E}_{\bar{\mathbf{t}}}(\Gamma)$ of the form

$$(C_1^* \oplus C_1^{**}), \ldots, (C_n^* \oplus C_n^{**}), (A \sqcap B)^*.$$

In general, a natural idea is to construct $\mathcal{D}_p$, the domain of order p, from the Herbrand's functions associated with formulas in $\Gamma = A_1, \ldots, A_n$ and to let the expansion of order p be of the form

$$(A_1^1 \oplus \ldots \oplus A_1^{q_1}), \ldots, (A_n^1 \oplus \ldots \oplus A_n^{q_n}),$$

where $A_i^1, \ldots, A_i^{q_i}$ result from A_{iH} by performing all possible substitutions of terms of $\mathcal{D}_p$ for the existential variables. The following counterexample shows that this form of Herbrand's Theorem fails in **MALL** even in the case of sequents of prenex formulas.

Counterexample. (3) Let Γ be

$$\bigvee y. \bigwedge x. P^\perp(x) \sqcup Q^\perp(y),\ \bigvee z. R^\perp(z),$$
$$\bigwedge y. \bigwedge z. \bigvee x_1. \bigvee x_2. (P(x_1) \otimes Q(y) \otimes R(z)) \sqcap (P(x_2) \otimes Q(z) \otimes R(y))$$

and let $\mathcal{E}_{\bar{\mathbf{t}}}(\Gamma)$ be the expansion over $\mathcal{D}_2$. Then $\vdash \Gamma$ is not derivable in **MALL**, but $\vdash \mathcal{E}_{\bar{\mathbf{t}}}(\Gamma)$ is, since

$$\vdash (P^\perp(\mathbf{x}[\mathbf{y}]) \sqcup Q^\perp(\mathbf{y})) \oplus (P^\perp(\mathbf{x}[\mathbf{z}]) \sqcup Q^\perp(\mathbf{z})),\ R^\perp(\mathbf{y}) \oplus R^\perp(\mathbf{z}),$$
$$(P(\mathbf{x}[\mathbf{y}]) \otimes Q(\mathbf{y}) \otimes R(\mathbf{z})) \sqcap (P(\mathbf{x}[\mathbf{z}]) \otimes Q(\mathbf{z}) \otimes R(\mathbf{y}))$$

is derivable in **MALL**.

Remark 3. V. C. V. de Paiva [1989] has defined the Dialectica Interpretation in the framework of Linear Logic.

Remark 4. In conclusion, for sequents of prenex formulas in the fragment **MLL** we have a very informative version of Herbrand's Theorem. It is unfortunate that we cannot refine this result to the case of non-prenex formulas, in which the number of arguments of the Herbrand's functions is minimized. If Linear Logic is to be used to make the constructive content of proofs explicit, then the additional information obtained through Linear Logic cannot be adequately expressed by these simple forms of Herbrand's Theorem.

10.2.1. An Application for Non Prenex Formulas.

The following result provides a way to extend Herbrand's analysis to some cases of non-prenex formulas in Direct Logic.

Definition. Let Γ be a multiset of arbitrary formula-occurrences in the language $\mathcal{L}$ of **DL**.

(i) Let **sb** be an assignment of the form $(A_1 \rightsquigarrow P_1, \ldots, A_n \rightsquigarrow P_n)$, with the following properties:

(1) for all $i \leq n$, A_i is a subformula of the formula occurrences in Γ, P_i is a propositional letter not occurring in Γ;

(2) for all $i, j \leq n$, if $i \neq j$ then P_i is a different letter from P_j, and A_i is a different formula-occurrence from A_j.

(ii) A *proof structure with prenex* $\forall$*-boxes* $(\mathbf{S}, \{\mathcal{R}_\forall\}, \mathbf{sb})$ with conclusion Γ is a proof-structure with $\forall$-boxes $(\mathbf{S}, \{\mathcal{R}_\forall\})$ with conclusion Γ (see section 3.5.2) together with an assignment $\mathbf{sb} = (A_1 \rightsquigarrow P_1, \ldots, A_n \rightsquigarrow P_n)$ satisfying the properties in (i), and in addition the following ones:

(1) For every structure $\mathcal{S}' \in \mathbf{S}$, only formula-occurrences in $\{A_1, \ldots, A_n\}$ occur in any non-logical axiom of $\mathcal{S}'$.

(2) For any $\mathcal{S}' \in \mathbf{S}$, let $A_{i_1}, \ldots, A_{i_n}$ be a list of all subformulas A_i occurring in a non-logical axiom of $\mathcal{S}'$ and let $\mathcal{S}'_{\mathbf{sb}}$ be the result of replacing every such A_{i_j} with P_{i_j} throughout $\mathcal{S}'$. Then the conclusions of $\mathcal{S}'_{\mathbf{sb}}$ are in prenex form.

(iii) Let **sb** be as in (i). Let Γ_H be the Herbrand form of Γ and let $\mathcal{L}'$ be the extension of $\mathcal{L}$ with the Herbrand functions of Γ_H. For some $\vec{\mathbf{t}}$ in $\mathcal{L}'$ let $\Gamma_H^{\vec{\mathbf{t}}}$ be an Herbrand expansion of Γ. Finally, let $\mathbf{sb}_H$ be the assignment $(A_1^* \rightsquigarrow P_1, \ldots, A_n^* \rightsquigarrow P_n)$, where A_i^* is the subformula in $\Gamma_H^{\vec{\mathbf{t}}}$ corresponding to A_i, i.e, the formula that results form A_i by applying the substitutions yielding $\Gamma_H^{\vec{\mathbf{t}}}$ from Γ.

(iv) Let $\mathcal{L}'$ and $\Gamma_H^{\vec{\mathbf{t}}}$ be as in (iii). Define a box $\mathcal{R}_H$ (relatively to $\Gamma_H^{\vec{\mathbf{t}}}$) by the condition

$$(*) \qquad \mathcal{R}_H(\overline{\Delta_H^{\vec{\mathbf{t}}}, A^*(\mathbf{s})}; \Delta_H^{\vec{\mathbf{t}}}, A^*(\mathbf{s}))$$

where $A^*(\mathbf{s})$ corresponds to an essentially universal subformula in Γ and $\mathbf{s}$ is a term of $\mathcal{L}'$ not occurring in $\Delta_H^{\vec{\mathbf{t}}}$.

(v) Given $\Gamma_H^{\vec{\mathbf{t}}}$ and $\mathbf{sb}_H$ as in (iii), a *proof-structure with prenex H-boxes* $(\mathbf{S}, \{\mathcal{R}_H\}, \mathbf{sb}_H)$ with conclusion $\Gamma_H^{\vec{\mathbf{t}}}$ is a proof-structure with H-boxes and with an assignment $\mathbf{sb}_H$ satisfying the following properties.

(1) For every structure $\mathcal{S}' \in \mathbf{S}$, only formula-occurrences in $\{A_1^*, \ldots, A_n^*\}$ occur in the non-logical axioms of $\mathcal{S}'$.

(2) For any $\mathcal{S}' \in \mathbf{S}$, let $\mathcal{S}'_{\mathbf{sb}_H}$ be the result of replacing every A_i^* that occurs in a non-logical axiom of $\mathcal{S}'$ with P_i throughout $\mathcal{S}'$. Then the conclusions of $\mathcal{S}'_{\mathbf{sb}_H}$ are Herbrand's expansions of formulas in prenex form.

Theorem. *Let* Γ *be a multiset of arbitrary first order formulas in a language* $\mathcal{L}$ *for* **DL**, *let* Γ_H *be the Herbrand form of* Γ *and let* $\mathcal{L}'$ *be the extension of* $\mathcal{L}$ *with the Herbrand functions of* Γ_H.

(i) *If* $\mathcal{S}$ *is a proof-net with conclusions* Γ, *then the procedure for Herbrand's Theorem in section* (10.2) *still yields terms* $\vec{\mathbf{t}} \in \mathcal{L}'$ *and a proof-net* $\mathcal{S}^*$ *with conclusions* $\Gamma_H^{\vec{\mathbf{t}}}$.

(ii) *Let* $(\mathbf{S}, \{\mathcal{R}_H\}, \mathbf{sb}_H)$ *be a proof-structure with prenex H-boxes and with conclusion* $\Gamma_H^{\vec{\mathbf{t}}}$, *for some* $\vec{\mathbf{t}} \in \mathcal{L}'$. *If* $(\mathbf{S}, \{\mathcal{R}_H\}, \mathbf{sb}_H)$ *is a proof-net, then from it we can construct a proof-net with conclusion* Γ.

Proof. (i) The procedure for the *only if* direction of Herbrand's Theorem for **DL** (section 10.2) does not depend on the fact that the formula in Γ are in prenex form. In particular, deleting one-premise links does not affect the vicious circle and connectedness conditions.

(ii) If $(\mathbf{S}, \{\mathcal{R}_H\}, \mathbf{sb}_H)$ is a proof-net, then each proof-structure $\mathcal{S}' \in \mathbf{S}$ is a proof-net, possibly with non-logical axioms. Moreover, each proof-structure $\mathcal{S}'_p = \mathcal{S}'\left[\begin{smallmatrix} A^*_{i_1} \\ P_{i_1} \end{smallmatrix}\right] \ldots \left[\begin{smallmatrix} A^*_{i_k} \\ P_{i_k} \end{smallmatrix}\right]$, after replacement of the formulas occurring in the non-nonlogical axioms with atomic letters, is such that its conclusions are Herbrand's expansions of prenex formulas. Thus the *if* direction of Herbrand's Theorem, section (10.2), applies to each $\mathcal{S}'_p$ and yields a proof-net $\mathcal{S}''_p$, possibly with non-logical axioms. Each proof-net $\mathcal{S}'' = \mathcal{S}''_p\left[\begin{smallmatrix} P_{i_1} \\ A_{i_1} \end{smallmatrix}\right] \ldots \left[\begin{smallmatrix} P_{i_k} \\ A_{i_k} \end{smallmatrix}\right]$, after replacement of the atomic letters occurring in the non-nonlogical axioms with the corresponding formulas in the assignment sb has

the property that its conclusions are subformulas of Γ. Let $\mathbf{S}'$ be the set of all the structures S'. The conditions on the box $\mathcal{R}_H$ guarantee that each relation

$$(*) \qquad \mathcal{R}_H(\overline{\Delta_H^{\bar{\mathbf{t}}}, A^*(\mathbf{s})}; \Delta_H^{\bar{\mathbf{t}}}, A^*(\mathbf{s}))$$

becomes a relation

$$(**) \qquad \mathcal{R}_\forall(\overline{\Delta, \bigwedge x.A}; \Delta, A(a)),$$

i.e., $(\mathbf{S}', \{\mathcal{R}_\forall\}, \mathbf{sb})$ is a a proof structure with prenex $\forall$-boxes. The elimination of the $\bigwedge$-boxes by the parameters condition is a consequence of the results in Section (4.6). ■

10.3. Π_2-Cut and Herbrand's Theorem in Linear Logic.

The Herbrand Theorem is defined for derivations (or proof-nets) in **MLL** *without Cut.* Let Π_2-*Cut* denote the Cut rule, restricted to Cut formulas in Π_2 form. We extend the definition of Herbrand expansion to certain sequent derivations containing Π_2-Cuts in **MLL**. We write Π_2-**Cut-MLL** for the sequent calculus **MLL** where the rule Cut is restricted to Π_2 formulas.

Let $\vdash \Gamma, A$ and $\vdash A^\perp, \Delta$ be sequents in **MLL** such that all formulas in Γ, Δ are in prenex form, and A is $\bigwedge x. \bigvee y.B$, with B quantifier-free.

Let $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$ be lists of all the Herbrand functions occurring in Γ_H and of Δ_H, respectively, and let $\mathbf{x}$ and $\mathbf{y}$ be the Herbrand functions occurring in A_H, $(A^\perp)_H$, respectively. Let $\mathcal{L}_1'$ $[\mathcal{L}_2']$ be the languages obtained extending $\mathcal{L}$ with $\mathbf{x}$ and the terms in $\vec{\mathbf{x}}$, [with $\mathbf{y}$ and the terms in $\vec{\mathbf{y}}$, respectively]. Also let $\mathcal{L}'$ be the extension of $\mathcal{L}$ by the terms in $\vec{\mathbf{x}}, \vec{\mathbf{y}}$.

Lemma. (i) *If the Herbrand theorem holds for* Γ, A *and* $A^\perp, \Delta$ *with expansions*

$$\Gamma_H^{\vec{\mathbf{r}}}, A_H^{\mathbf{s}} \qquad \textit{and} \qquad (A^\perp)_H^{\mathbf{t}}, \Delta_H^{\vec{\mathbf{u}}},$$

respectively, where $\vec{\mathbf{r}}, \mathbf{s}$ *belong to* $\mathcal{L}_1'$ *and* $\mathbf{t}, \vec{\mathbf{u}}$, *belong to* $\mathcal{L}_2'$, *then the Herbrand Theorem holds for* Γ, Δ, *with expansion*

$$\Gamma_H^{\vec{\mathbf{r}}'}, \Delta_H^{\vec{\mathbf{u}}'},$$

for some lists $\vec{\mathbf{r}}'$ *and* $\vec{\mathbf{u}}'$ *of terms of* $\mathcal{L}'$.

(ii) *Let* $\mathcal{S}_1$ *and* $\mathcal{S}_2$ *be proof-nets for* **MLL** *with conclusions* Γ, A *and* $A^\perp, \Delta$ *and without cuts, and let* $\mathcal{S}_0$ *be the result of applying the cut-elimination procedure to the cut* $\underline{A \otimes A^\perp}$ *in* $\mathcal{S}_1\, {}_{A \otimes A^\perp}\, \mathcal{S}_2$. *Let* $\Gamma_H^{\vec{\mathbf{r}}'}, \Delta_H^{\vec{\mathbf{u}}'}$ *be the Herbrand expansion given by* (i) *and let* $\Gamma_H^{\vec{\mathbf{r}}''}, \Delta_H^{\vec{\mathbf{u}}''}$ *be the Herbrand expansion given by the Herbrand Theorem in section* (10.2) *applied to* $\mathcal{S}_0$. *Then* $\vec{\mathbf{r}}'$ *coincides with* $\vec{\mathbf{r}}''$ *and* $\vec{\mathbf{u}}'$ *coincides with* $\vec{\mathbf{u}}''$.

Proof. (i) We have

$$A_H^{\mathbf{s}} = B(\mathbf{x}, \mathbf{s}), \qquad \text{and} \qquad (A^\perp)_H^{\mathbf{t}} = B^\perp(\mathbf{t}, \mathbf{y}[\mathbf{t}]).$$

where $\mathbf{s}$ is a term of $\mathcal{L}_1'$. We write $t\left[{y \atop z}\right]$ or $X\left[{y \atop z}\right]$ for the result of substituting z for every occurrence of y in t or X. If $\vec{t} = t_1, \ldots, t_n$, then $\vec{t}\left[{y \atop z}\right]$ is the list $t_1\left[{y \atop z}\right], \ldots, t_n\left[{y \atop z}\right]$. We write $\lambda\mathbf{x}.\mathbf{s}[\mathbf{x}]$ with the obvious meaning, and if $\mathbf{x}$ does not occur in $\mathbf{s}$, then $\lambda\mathbf{x}.\mathbf{s}[\mathbf{x}] = \mathbf{s}$.

Consider $\vec{\mathbf{r}}'$, $\mathbf{s}'$, $\mathbf{t}$ and $\vec{\mathbf{u}}'$ obtained by the following steps:

(I) Let $\vec{\mathbf{u}}' = \vec{\mathbf{u}}\left[{\lambda\mathbf{x}.\mathbf{y}[\mathbf{x}] \atop \lambda\mathbf{x}.\mathbf{s}[\mathbf{x}]}\right]$ and $\mathbf{t}' = \mathbf{t}\left[{\lambda\mathbf{x}.\mathbf{y}[\mathbf{x}] \atop \lambda\mathbf{x}.\mathbf{s}[\mathbf{x}]}\right]$.

(II) Let $\vec{\mathbf{r}}' = \vec{\mathbf{r}}\left[{\mathbf{x} \atop \mathbf{t}'}\right]$ and $\mathbf{s}' = \vec{s}\left[{\mathbf{x} \atop \mathbf{t}'}\right]$.

Clearly the terms $\mathbf{s}'$, $\mathbf{t}'$ and those in $\vec{\mathbf{r}}'$, $\vec{\mathbf{u}}'$ belong to $\mathcal{L}'$. We claim that Herbrand's Theorem holds for Γ, Δ with expansion $\Gamma_H^{\vec{\mathbf{r}}'}, \Delta_H^{\vec{\mathbf{u}}'}$.

Suppose $\mathcal{S}_1$, $\mathcal{S}_2$ are proof-nets with conclusions Γ, A and $A^\perp, \Delta$, respectively, and that $\mathcal{S}_1'$, $\mathcal{S}_2'$ are the proof-nets with conclusions $\Gamma_H^{\vec{\mathbf{r}}}, A_H^{\mathbf{s}}$ and $(A^\perp)_H^{\mathbf{t}}, \Delta_H^{\vec{\mathbf{u}}}$, respectively, given by Herbrand's Theorem. Replace $\lambda\mathbf{x}.\mathbf{s}[\mathbf{x}]$ for $\lambda\mathbf{x}.\mathbf{y}[\mathbf{x}]$ everywhere in $\mathcal{S}_2$; replace $\mathbf{t}'$ for $\mathbf{x}$ everywhere in $\mathcal{S}_1$. We obtain proof-nets $\mathcal{S}_1''$, $\mathcal{S}_2''$ with conclusions $\Gamma_H^{\vec{\mathbf{r}}'}, A^*$ and $(A^*)^\perp, \Delta_H^{\vec{\mathbf{u}}'}$, where $A^* = B(\mathbf{t}', \mathbf{s}[\mathbf{t}'])$. In conclusion, $\mathcal{S}_1''\, {}_{A^*} \otimes_{(A^*)^\perp}\, \mathcal{S}_2''$ is a proof-net in propositional **MLL** with conclusion $\Gamma_H^{\vec{\mathbf{r}}'}, \Delta_H^{\vec{\mathbf{u}}'}$.

Conversely, if we have a proof-net $\mathcal{S}_1''\, {}_{A^*} \otimes_{(A^*)^\perp}\, \mathcal{S}_2''$ with conclusion $\Gamma_H^{\vec{\mathbf{r}}'}, \Delta_H^{\vec{\mathbf{u}}'}$ and cut $\underline{A^* \otimes A^{*\perp}}$ for propositional **MLL**, then we can apply the cut-elimination theorem and obtain a cut-free proof-net $\mathcal{S}_0'$ with the same conclusion. Now the Herbrand Theorem in section (10.2) applied to $\mathcal{S}_0'$ yields a proof-net with conclusion Γ, Δ.

(ii) Consider $\mathcal{S} = \mathcal{S}_1 \; {}_A\!\otimes_{A^\perp} \mathcal{S}_2$.

$$\dfrac{\Gamma \quad \dfrac{\dfrac{\begin{matrix}\mathcal{S}_1\\ B(a,s)\end{matrix}}{\bigvee y.B(a,y)}}{\bigwedge x.\bigvee y.B(x,y)} \quad \dfrac{\dfrac{\begin{matrix}\mathcal{S}_2\\ B^\perp(t,b)\end{matrix}}{\bigwedge y.B^\perp(t,y)}}{\bigvee x.\bigwedge y.B^\perp(t,y)} \quad \Delta}{\underline{A \otimes A^\perp}}$$

Let $\vec{a} * \langle a \rangle$ [or $\langle b \rangle * \vec{b}$] be lists of all the eigenvariables occurring in $\mathcal{S}_1$ [or $\mathcal{S}_2$]. Notice that b does not occur in t, since b occurs only inside the empire of $B(t,b)$. Let $\vec{r}$ [or $\vec{u}$] be a list of all the terms w different from s [or t] such that if w occurs in $\vec{r}$ [or in $\vec{u}$], then for some C

$$\frac{C(w)}{\bigvee x.C(x)}$$

is a link of $\mathcal{S}_1$ [or of $\mathcal{S}_2$]. We assume that Γ, A and $A^\perp, \Delta$ consist of closed formulas, thus no eigenvariable occurs in A. Hence in $\mathcal{S}$ there are no eigenvariables c, d such that c occurs in $\mathcal{S}_1$, d occurs in $\mathcal{S}_2$ and either $c <^0 d$ or $d <^0 c$.

The first two steps of the cut-elimination procedure involve the following substitutions:

(1) replace t for a everywhere in $\mathcal{S}_1$. We write $\mathcal{S}_1\left[{a \atop t}\right]$ for the resulting substructure. Let $\vec{r}\,' = \vec{r}\left[{a \atop t}\right]$, $s' = s\left[{a \atop t}\right]$.

(2) replace s' for b everywhere in $\mathcal{S}_2$. We write $\mathcal{S}_2\left[{b \atop s'}\right]$ for the resulting substructure. Let $\vec{u}\,' = \vec{u}\left[{b \atop s'}\right]$.

Let $\mathcal{S}^*$ be the proof-net resulting from the two step in question. Notice that the other steps of the cut-elimination procedure involve only propositional reductions, since B is quantifier-free and that in **MLL** propositional steps do not affect the lists $\vec{r}\,', \vec{u}\,'$.

Fact 4. *In $\mathcal{S}^*$, if d is any eigenvariable occurring in t and e is an eigenvariable in $\vec{a}$, then we cannot have $d <_t e$.*

Otherwise, suppose $d <^0 c_1 <^0 \ldots <^0 c_n <^0 e$, where $d, c_1, \ldots, c_n$ are all in $\vec{b}$. Since all formulas in Γ, Δ are prenex, it follows that if c, e are eigenvariables and c is associated with the quantifier $\bigwedge x.C$ then $c <^0 e$ only if e occurs in $\bigwedge x.C$. From this fact we may conclude that $d <^0 c_1 <^0 \ldots <^0 c_n$ also in $\mathcal{S}$, and furthermore,

that in $\mathcal{S}^*$ the eigenvariable e occurs in $\bigwedge x.C_n$, where c_n is associated with $\bigwedge x.C_n$. Since $\bigwedge x.C_n$ occurs in $\mathcal{S}_2\left[{}^{b}_{s'}\right]$, e must occur in s'. By the Fact in the proof of the Reduction Lemma, $\bigwedge$-case, section (7.3), in $\mathcal{S}$ we must have $c_n <_t b$ and since d occurs in t, $b <^0 d$. Thus in $\mathcal{S}$ we have

$$d <^0 c_1 <^0 \ldots <^0 c_n <_t b <^0 d,$$

a contradiction.

Fact 5. *Let e, d be any eigenvariables in $\vec{a}$, $\vec{b}$, respectively. Suppose that there is an occurrence of e in $\vec{r}\,'$ or in s' and that $e <_t d$ in $\mathcal{S}^*$. Then there are eigenvariables e_0, d_0 and a link*

$$\frac{C(t)}{\bigvee x.C(x)}$$

such that in $\mathcal{S}^*$

(i) $e_0 <^0 d^0$ *and* $e \leq_t e_0$ *and* $d_0 \leq_t d$ (where " $x \leq_t y$ " means " $x = y$ *or* $x <_t y$ ");

(ii) d_0 *occurs in* t;

(iii) *if* e_0 *is associated with* $\bigwedge x.E_0$, *then* $\bigwedge x.E_0 \prec C(t)$.

Since $e <_t d$, consider the first e_0, d_0 (starting from e) such that $e \leq_t e_0 <^0 d_0 \leq_t d$, and moreover $e, \ldots, e_0$ belong to $\vec{a}$ and d_0 belongs to $\vec{b}$. Since all formulas in Γ are prenex, d_0 must occur in $E_0(e_0)$ and since E_0 occurs in $\mathcal{S}_1\left[{}^{a}_{t}\right]$, it follows that d_0 occurs in t. Since no eigenvariable occurs in Γ, Δ, there must be a $\bigvee$-link satisfying (iii).

A similar argument shows that

Fact 6. *Let e, d be any eigenvariables in $\vec{a}$, $\vec{b}$, respectively. Suppose that there is an occurrence of d in $\vec{u}$ such that $d <_t e$ in $\mathcal{S}^*$. Then there are eigenvariables e_0, d_0 and a link*

$$\frac{C(s')}{\bigvee x.C(x)}$$

such that in $\mathcal{S}^*$

(i) $d_0 <^0 e^0$ *and* $d \leq_t d_0$ *and* $e_0 \leq_t e$;

(ii) e_0 *occurs in* s';

(iii) *if* d_0 *is associated with* $\bigwedge x.D_0$, *then* $\bigwedge x.D_0 \prec C(s')$.

Consider now the procedure for Herbrand's Theorem for **MLL** applied to $\mathcal{S}^{*}$[1] It produces a pair of lists

$$[0]_{\mathcal{S}^\bullet} \quad \mathbf{v}_1[w_{1,1},\dots,w_{1,k_1}],\dots,\mathbf{v}_h[w_{h,1},\dots,w_{h,k_h}]; \qquad r'_1,\dots,r'_p,u'_1,\dots,u'_q,s',t.$$

where $\mathbf{v}_i[w_{i,1},\dots,w_{i,k_1}]$ is either $\mathbf{x}_j[w_{i,1},\dots,w_{i,k_i}]$ or $\mathbf{y}_j[w_{i,1},\dots,w_{i,k_i}]$, and in h steps it yields a pair of lists

$$(\star)_{\mathcal{S}^\bullet} \quad \mathbf{v}_1[\mathbf{w}_{1,1},\dots,\mathbf{w}_{1,k_1}],\dots,\mathbf{v}_h[\mathbf{w}_{h,1},\dots,\mathbf{w}_{h,k_h}]; \qquad \mathbf{r}''_1,\dots,\mathbf{r}''_p,\mathbf{u}''_1,\dots,\mathbf{u}''_q,\mathbf{s}'',\mathbf{t}''.$$

Write $\vec{\mathbf{r}}''$ for $\mathbf{r}''_1,\dots,\mathbf{r}''_p$ and $\vec{\mathbf{u}}''$ for $\mathbf{u}''_1,\dots,\mathbf{u}''_q$.

We need to compare $\vec{\mathbf{r}}''$, $\mathbf{s}''$, $\mathbf{t}''$, $\vec{\mathbf{u}}''$ with the terms $\vec{\mathbf{r}}'$, $\mathbf{s}'$, $\mathbf{t}'$, $\vec{\mathbf{u}}'$ obtained from the procedure described in part (i). First Herbrand's Theorem for **MLL** is applied to the cut-free proof-nets $\mathcal{S}_1$ and $\mathcal{S}_2$. For $\mathcal{S}_1$ the procedure produces a pair of lists

$$[0]_{\mathcal{S}_1} \qquad \mathbf{x},\mathbf{x}_1[r_{1,1},\dots,r_{1,k_1}],\dots,\mathbf{x}_m[r_{m,1},\dots,r_{m,k_m}]; \qquad r_1,\dots,r_p,s,$$

and in m steps it yields a pair of lists

$$(\star)_{\mathcal{S}_1} \qquad \mathbf{x},\mathbf{x}_1[\mathbf{r}_{1,1},\dots,\mathbf{r}_{1,k_1}],\dots,\mathbf{x}_m[\mathbf{r}_{m,1},\dots,\mathbf{r}_{m,k_m}]; \qquad \mathbf{r}_1,\dots,\mathbf{r}_p,\mathbf{s}.$$

We let $\vec{\mathbf{r}}$ be $\mathbf{r}_1,\dots,\mathbf{r}_p$. Similarly, for $\mathcal{S}_2$ we obtain

$$[0]_{\mathcal{S}_2} \qquad \mathbf{y}_1[u_{1,1},\dots,u_{1,k_1}],\dots,\mathbf{y}[t],\dots,\mathbf{y}_\ell[u_{\ell,1},\dots,r_{\ell,k_\ell}]; \qquad u_1,\dots,u_q,t,$$

and in ℓ steps

$$(\star)_{\mathcal{S}_2} \qquad \mathbf{y}_1[\mathbf{u}_{1,1},\dots,\mathbf{u}_{1,k_1}],\dots,\mathbf{y}[\mathbf{t}],\dots,\mathbf{y}_\ell[\mathbf{u}_{\ell,1},\dots,\mathbf{u}_{\ell,k_\ell}]; \qquad \mathbf{u}_1,\dots,\mathbf{u}_q,\mathbf{t}.$$

We let $\vec{\mathbf{u}}$ be $\mathbf{u}_1,\dots,\mathbf{u}_q$. The terms $\vec{\mathbf{r}}'$, $\mathbf{s}'$, $\mathbf{t}'$, $\vec{\mathbf{u}}'$ are obtained from $\vec{\mathbf{r}}$, $\mathbf{s}$, $\mathbf{t}$, $\vec{\mathbf{u}}$ by the substitutions (I) and (II) described in part (i).

We can obtain more information about $\vec{\mathbf{r}}'$, $\mathbf{s}'$, $\mathbf{t}'$, $\vec{\mathbf{u}}'$ from the following remark. Notice that in the derivation

$$\dfrac{\begin{matrix}\mathcal{S}_2^-\\ B^\perp(t,b)\end{matrix}}{\bigwedge y.B^\perp(t,y)}\ \Delta$$

[1] *Notice that in* $\mathcal{S}^\bullet$ *the only cut is a quantifier-free formula.*

the Herbrand function associated with $\bigwedge y$ is $\mathbf{y}$ and that Herbrand's Theorem holds for $\bigwedge y.B^{\perp}(t,y), \Delta$ with the expansion $(B^{\perp})_H^{\mathbf{t}'''}, \Delta_H^{\vec{\mathbf{u}}'''}$ such that $\mathbf{y}$ does not occur in $\mathbf{t}'''$.

Indeed, the procedure for Herbrand's Theorem applied to $\mathcal{S}_2^-$ produces the pair of list

$$[0]_{\mathcal{S}_2^-} \qquad \mathbf{y}, \mathbf{y}_1[u_{1,1},\ldots,u_{1,k_1}],\ldots,\mathbf{y}_\ell[u_{\ell,1},\ldots,r_{\ell,k_\ell}]; \qquad u_1,\ldots,u_q,t,$$

and in ℓ steps

$$(\star)_{\mathcal{S}_2^-} \qquad \mathbf{y}, \mathbf{y}_1[\mathbf{u}'''_{1,1},\ldots,\mathbf{u}'''_{1,k_1}],\ldots,\mathbf{y}_\ell[\mathbf{u}'''_{\ell,1},\ldots,\mathbf{u}'''_{\ell,k_\ell}]; \qquad \mathbf{u}'''_1,\ldots,\mathbf{u}'''_q,\mathbf{t}'''.$$

We write $\vec{\mathbf{u}}'''$ for $\mathbf{u}'''_1,\ldots,\mathbf{u}'''_q$. Suppose $\mathbf{y}$ occurs in $\mathbf{t}'''$. Then there is a sequence of substitutions

$$\left(\cdots\left(t\begin{bmatrix} b_{i_1} \\ \mathbf{y}_{i_1}[u_{i_1,1},\ldots,u_{i_1,k_{i_1}}]\end{bmatrix}\right)\begin{bmatrix} b_{i_2} \\ \mathbf{y}_{i_2}[u_{i_2,1},\ldots,u_{i_2,k_{i_2}}]\end{bmatrix}\cdots\right)\begin{bmatrix} b \\ \mathbf{y}\end{bmatrix}$$

and by Fact 1 in section (10.2) $b >_t \ldots >_t b_{i_2} >_t b_{i_1}$. Since b_{i_1} occurs both in $B^{\perp}(t,b)$ and in $\bigwedge y.B^{\perp}(t,y)$, it follows that $b_{i_1} >^0 b$, thus $b >_t b_{i_1} >^0 b$, a contradiction.

Moreover, in **MLL** we may assume that eigenvariables and Herbrand's functions associated with different $\bigwedge$-links are distinct, therefore we may let[2]

$$\begin{aligned} \mathbf{t} &= \mathbf{t}'''\begin{bmatrix}\mathbf{y}\\ \mathbf{y}[\mathbf{t}]\end{bmatrix} = \mathbf{t}''', \\ \vec{\mathbf{u}} &= \vec{\mathbf{u}}'''\begin{bmatrix}\mathbf{y}\\ \mathbf{y}[\mathbf{t}]\end{bmatrix} \quad . \end{aligned}$$

In conclusion, we obtain

$$\begin{array}{llll} \mathbf{t}' & = \mathbf{t}\left[{}^{\lambda \mathbf{x}.\mathbf{y}[\mathbf{x}]}_{\lambda \mathbf{x}.\mathbf{s}[\mathbf{x}]}\right] & \text{(by I)} & = \mathbf{t}; \\ \mathbf{s}' & = \mathbf{s}\left[{}^{\mathbf{x}}_{\mathbf{t}'}\right] & \text{(by II)} & = \mathbf{s}\left[{}^{\mathbf{x}}_{\mathbf{t}}\right]; \\ \vec{\mathbf{u}}' & = \vec{\mathbf{u}}\left[{}^{\lambda \mathbf{x}.\mathbf{y}[\mathbf{x}]}_{\lambda \mathbf{x}.\mathbf{s}[\mathbf{x}]}\right] & \text{(by I)} & = \vec{\mathbf{u}}\left[{}^{\mathbf{y}[\mathbf{t}]}_{\mathbf{s}'}\right]; \\ \vec{\mathbf{r}}' & = \mathbf{r}\left[{}^{\mathbf{x}}_{\mathbf{t}'}\right] & \text{(by II)} & = \vec{\mathbf{r}}\left[{}^{\mathbf{x}}_{\mathbf{t}}\right]; \end{array}$$

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[2] Notice that in the case of a logic with Contraction, e.g., in **MALL**, if $\bigvee x.\bigwedge y.B^{\perp}(t,y)$ was the conclusion of a Contraction link, then a term of the form $\mathbf{y}[\mathbf{w}]$, may occur in $(A^{\perp})_H^{\mathbf{t}}, \Delta_H^{\vec{\mathbf{u}}}$ for some $\mathbf{w}$ different from $\mathbf{t}$ — in fact $\mathbf{y}[\mathbf{w}]$ may occur in $\mathbf{t}$.

We conclude the proof of the Lemma and show that the terms $\vec{\mathbf{r}}''$, $\mathbf{s}''$, $\mathbf{t}''$, $\vec{\mathbf{u}}''$, which are obtained by applying Herbrand's Theorem after eliminating the (first order) cut in $\mathcal{S}$, coincide with the terms $\vec{\mathbf{r}}'$, $\mathbf{s}'$, $\mathbf{t}'$, $\vec{\mathbf{u}}'$, which are obtained by performing the substitutions described in part (i) after applying Herbrand's Theorem separately to $\mathcal{S}_1$ and in $\mathcal{S}_2$.

By Fact 4, if the eigenvariables d, d_1 are such that d occurs in t and $d <_t d_1$ in $\mathcal{S}^*$, then d_1 is in $\vec{b}$. Therefore we may conclude that $d <_t d_1$ in $\mathcal{S}^*$ if and only if $d <_t d_1$ in $\mathcal{S}_2$. Consider the list of universal terms in $[0]_{\mathcal{S}^*}$: it follows that only Herbrand's functions of the form $\mathbf{y}_i[x_{i,1}, ..., x_{i,k_i}]$, may occur in any chain of substitutions beginning with the replacement of some eigenvariable in t, by Fact 1. Moreover, the ordering of the terms $\mathbf{y}_i[w_{i,1}, ..., w_{i,k_i}]$ in $[0]_{\mathcal{S}^*}$ and of the terms $\mathbf{y}_i[u_{i,1}, ..., u_{i,k_i}]$ in $[0]_{\mathcal{S}_2}$ both preserve the same relations $d <_t d_1$ between the eigenvariables $\vec{b}$ in $\mathcal{S}^*$ and in $\mathcal{S}_2$. Therefore by Fact 2 the term $\mathbf{t}''$ in $(\star)_{\mathcal{S}^*}$ coincides with the term $\mathbf{t}$ in $(\star)_{\mathcal{S}_2}$. But $\mathbf{t}' = \mathbf{t}$, thus $\mathbf{t}' = \mathbf{t}''$.

If e', e'' are eigenvariables in $\vec{a}$, then in $\mathcal{S}^*$, by Facts 4 and 5 there is no $d \in \vec{b}$ such that $e' <_t d <_t e''$. Also, there is no $e \in \vec{a}$ such that $a <^0 e$ in $\mathcal{S}_1$. Therefore using the Fact in the proof of the Reduction Lemma, $\bigwedge$-case, section (7.3), we may conclude that $e' <_t e''$ in $\mathcal{S}^*$ if and only if $e' <_t e''$ in $\mathcal{S}_1$. Let $\mathbf{x}'[v'_1, \ldots, v'_{k'}]$ and $\mathbf{x}''[v''_1, \ldots, v''_{k''}]$ be the Herbrand functions associated with e', e'', respectively. Then by arguing like in the previous paragraph we conclude that $\mathbf{x}''$ occurs in the term $\mathbf{x}'[\mathbf{w}'_1, \ldots, \mathbf{w}'_{k'}]$ of $(\star)_{\mathcal{S}^*}$ if and only if $\mathbf{x}''$ occurs in the term $\mathbf{x}'[\mathbf{r}'_1, \ldots, \mathbf{r}'_{k'}]$ of $(\star)_{\mathcal{S}_1}$ Similarly, $\mathbf{x}''$ occurs in the term $\mathbf{s}''$ of $(\star)_{\mathcal{S}^*}$ if and only if $\mathbf{x}''$ occurs in the term $\mathbf{s}$ of $(\star)_{\mathcal{S}_1}$.

Moreover, if $\mathbf{y}'[z_1, \ldots, z_h]$ is associated with any eigenvariable $d \in \vec{b}$ and $\mathbf{y}'$ occurs in the term $\mathbf{x}'[\mathbf{w}'_1, \ldots, \mathbf{w}'_{k'}]$ of $(\star)_{\mathcal{S}^*}$ then $e' <_t d$ by Fact 1. It follows from Fact 5 that $\mathbf{y}'$ occurs only in $\mathbf{t}$ and that $\mathbf{t}$ is a subterm of $\mathbf{x}'[\mathbf{w}'_1, \ldots, \mathbf{w}'_{k'}]$. Similarly, any occurrence of $\mathbf{y}'$ in $\mathbf{s}''$ is within the subterm $\mathbf{t}$ of $\mathbf{s}''$.

Finally, for any occurrence of the Herbrand function $\mathbf{x}$ in any term $\mathbf{x}'[\mathbf{r}'_1, \ldots, \mathbf{r}'_{k'}]$ of $(\star)_{\mathcal{S}_1}$, there is an occurrence of $\mathbf{t}$ in the term $\mathbf{x}'[\mathbf{w}'_1, \ldots, \mathbf{w}'_{k'}]$ of $(\star)_{\mathcal{S}^*}$, since $\mathbf{x}$ corresponds to the eigenvariable a of $\mathcal{S}_1$ and t is replaced for a in $\mathcal{S}^*$. Similarly,

any occurrence of $\mathbf{x}$ in the term $\mathbf{s}$ of $(\star)_{S_1}$, corresponds to an occurrence of $\mathbf{t}$ in the term $\mathbf{s}''$ of $(\star)_{S^\bullet}$. We conclude that

$$\mathbf{s}'' = \mathbf{s}\begin{bmatrix}\mathbf{x}\\ \mathbf{t}\end{bmatrix} = \mathbf{s}'.$$

The argument for $\vec{\mathbf{r}}'' = \vec{\mathbf{r}}'$ is similar. To show $\vec{\mathbf{u}}'' = \vec{\mathbf{u}}'$ we argue in the same way, using Fact 6 instead of Fact 5. ∎

Despite all the restrictions, the above result is useful, since it allows us to apply the Herbrand Theorem to the derivations of the premises of certain Cuts, without losing any information.

10.3.1. Π_2-Induction in Linear Logic.

Definition. Let Π_2-**IR** be the induction rule

$$\frac{\vdash A(0) \qquad \vdash A^\perp(n), A(n+1)}{\vdash A(t)}$$

where $A(z)$ is in Π_2 form and t is any term. Let Π_2-**Cut**-**IR**-**MLL** be **MLL** with the addition of the Π_2-**IR**, where the rule of Cut is restricted to formulas in Π_2-form.

Remark. If in the conclusion $A(t)$ of an application of the Π_2-Induction Rule the term t is a numeral, then the induction rule can be replaced by t applications of Cut.

Lemma. (i) *Let* $\vdash A(0)$ *and* $\vdash A^\perp(n), A(n+1)$ *be sequents the language* $\mathcal{L}$ *for* **MLL** *such that* $A(z)$ *is of the form* $\bigwedge x.\bigvee y.B(x,y,z)$*, with* B *quantifier-free. Let Herbrand's Theorem hold for both sequents with Herbrand's expansions*

$$B(\mathbf{x}, \mathbf{s}_0, 0) \quad \text{and} \quad B(\mathbf{t}_n, \mathbf{y}[\mathbf{t}_n], n), B(\mathbf{x}, \mathbf{s}_{n+1}, n+1),$$

respectively. Define

$$\begin{aligned} \mathbf{f}(0,\mathbf{x}) &= \mathbf{s}_0;\\ \mathbf{f}(n+1,\mathbf{x}) &= \mathbf{s}_{n+1}\begin{bmatrix}\lambda\mathbf{x}.\mathbf{y}[\mathbf{x}]\\ \lambda\mathbf{x}.\mathbf{f}(n,\mathbf{x})\end{bmatrix}.\end{aligned}$$

Then for each number n*, the Herbrand Theorem holds for* $\vdash A(n)$ *with Herbrand's expansion* $B(\mathbf{x}, \mathbf{f}(n,\mathbf{x}), n)$.

(ii) *Let $\mathcal{D}$ be a derivation in* Π_2-**Cut-IR-MLL** *of a sequent* $\vdash \Gamma$, *such that all formulas of* Γ *are closed and prenex. In addition suppose that the term* t *in each conclusion* $A(t)$ *of the Induction Rule is a numeral. Let* $\mathcal{D}_1$ *result from* $\mathcal{D}$ *by eliminating all occurrences of Cut and Induction. Moreover, let* $\mathcal{L}'$ *be the extensions of* $\mathcal{L}$ *with the Herbrand functions for the universal quantifiers occurring in* $\mathcal{D}_1$ *and let* $\mathcal{L}''$ *be the extensions of* $\mathcal{L}$ *with the Herbrand functions for the universal quantifiers occurring in* $\mathcal{D}$ *and the scheme of definition by primitive recursion. Then there are terms* $\vec{\mathbf{t}}'$ *in* $\mathcal{L}'$ *and term* $\vec{\mathbf{t}}''$ *in* $\mathcal{L}''$ *such that Herbrand's Theorem holds with expansions* $\Gamma_H^{\vec{\mathbf{t}}'}$ *and* $\Gamma_H^{\vec{\mathbf{t}}''}$ *and for each* i and $\mathbf{t}'_i \in \vec{\mathbf{t}}'$, $\mathbf{t}''_i \in \vec{\mathbf{t}}''$

$$\mathbf{t}'_i = \mathbf{t}''_i.$$

(iii) *Let $\mathcal{D}$ be a derivation in* Π_2-**Cut-IR-MLL** *of a sequent* $\vdash \Gamma$, *such that all formulas of* Γ *are closed and prenex. Let* $\vec{\mathbf{y}}$ *be a list of all Herbrand's functions for the universal quantifiers occurring in* $\mathcal{D}$ *and let* $\mathcal{L}'$ *be the language* $\mathcal{L}$, *extended with* $\vec{\mathbf{y}}$ *and the scheme of definition by primitive recursion. Define* $\Gamma_H^{\vec{\mathbf{t}}}$ *as in* (i), *where the terms in* $\vec{t}$ *belong to* $\mathcal{L}'$. *Then* $(\Gamma_H^{\vec{\mathbf{t}}}\left[\frac{\vec{y}}{\vec{s}}\right])$ *is true for any choice of (appropriate) numerical functions and constants* $\vec{s}$ *for* $\vec{\mathbf{y}}$.

Proof. (i) and (ii) follow from (i) and (ii) of the Lemma in section (10.3) and the above Remark. (iii) follows from (ii). (iii) defines a rudimentary *functional interpretation* for the fragment of Peano Arithmetic under consideration. ■

Remark. Let $\frac{C(t)}{Qx.C}$ be a quantificational link in a proof-net $\mathcal{S}$. We say that x is *traceable* to a (logical) axiom if there is no proof-net $\mathcal{S}'$ such that t does not occur in any axiom of $\mathcal{S}'$ and $\mathcal{S}'$ results from $\mathcal{S}$ only by renaming eigenvariables and terms in links different from $\frac{C(t)}{Qx.C}$. It is easy to see that *the Lemma can be extended to sequent derivations in Direct Logic with* Π_2-**Cut** *and* Π_2-**IR**, *if for every quantifier* $Qx.C$ *occurring a Cut or Induction formula, the occurrences of* x *in both premises of the Cut or Induction can be traced to some axiom.*

10.4. Functionals of Classical Predicate Calculus.

The system obtained by adding the unrestricted Contraction Rule to the sequent calculus for $\mathbf{LD}^+$ clearly is simply (a reformulation of) $\mathbf{LK}$. Suppose in the

given language **L** conditional terms `if ... then ... else ...` are definable. Consider an instance of Contraction

$$\frac{\vdash \Gamma, A, A}{\vdash \Gamma, A}$$

and let $\Gamma_H^{\vec{\mathbf{r}}}, A_H^{\vec{\mathbf{s}}}, A_H^{\vec{\mathbf{t}}}$ be an Herbrand expansion for the upper sequent. Let $\vec{\mathbf{x}}$ and $\vec{\mathbf{x}}'$ be the Herbrand functions occurring in the two occurrences of A_H: clearly if $\vec{\mathbf{x}} = \mathbf{x}_1, \ldots, \mathbf{x}_k$ and $\vec{\mathbf{x}}' = \mathbf{x}'_1, \ldots, \mathbf{x}'_k$, then $\mathbf{x}_i$ and $\mathbf{x}'_i$ have the same arity and can be identified. Also $\vec{\mathbf{s}} = \mathbf{s}_1, \ldots, \mathbf{s}_m$ and $\vec{\mathbf{t}} = \mathbf{t}_1, \ldots, \mathbf{t}_m$. Define $\vec{\mathbf{u}} = \mathbf{u}_1, \ldots, \mathbf{u}_n$ by letting for each $i \leq n$

$$\mathbf{u}_i = \texttt{if } A_H^{\vec{\mathbf{s}}} \texttt{ then } \mathbf{s}_i \texttt{ else } \mathbf{t}_i.$$

Let $\Gamma_H^{\vec{\mathbf{r}}}, A_H^{\vec{\mathbf{u}}}$ be the Herbrand expansion for the lower sequent. We can show by a semantic argument that Herbrand's Theorem holds for $\Gamma_H^{\vec{\mathbf{r}}}, A_H^{\vec{\mathbf{s}}}, A_H^{\vec{\mathbf{t}}}$ if and only if it holds for $\Gamma_H^{\vec{\mathbf{r}}}, A_H^{\vec{\mathbf{u}}}$.

10.5. No Counterexample Interpretation.

The Herbrand theorem fails if **LK** is extended by the unrestricted induction rule: we need a functional interpretation, the No Counterexample Interpretation (NCI). Let $\mathcal{L}$ be the first order language of Peano Arithmetic. If $S : \Gamma \vdash \Delta$ is any sequent in $\mathcal{L}$, let $\mathcal{L}'$ be **L** extended by the Herbrand functions for S as above. Then the language $\mathcal{L}_F$ of the NCI is $\mathcal{L}'$ extended by functionals of type ≤ 2: $\lambda \mathbf{y}_1 \ldots \mathbf{y}_m . X(\mathbf{y}_1, \ldots, \mathbf{y}_m)$, for the essentially existential quantifiers of S. The functionals are defined by α-recursion, for ordinals $\alpha < \epsilon_0$ (*extended Grzegorczyk hierarchy*) (See Rose [1984]). Consider the extended Ackermann functions

$$\begin{aligned} f_1(x) &= 2x \\ f_{\alpha+1}(x) &= f_\alpha^{(x)}(x), \\ f_\alpha(x) &= f_{\alpha(x)}(x), \qquad \text{for } \alpha \text{ limit.} \end{aligned}$$

where $f^{(x)}$ denotes the x-th iteration of f and where $\alpha(x)$ is the x-th element in the *natural sequence* for α, defined as follows.

If

$$\alpha = n_1\omega^{\alpha_1} + \ldots + n_t\omega^{\alpha_t}$$

is in Cantor normal form, then $\alpha(x)$ is inductively defined by

$$\alpha(x) = n_1\omega^{\alpha_1} + \ldots + n_{t-1}\omega^{\alpha_{t-1}} + (n_t - 1)\omega^{\alpha_t} + x\omega^{\beta_t} \qquad \text{if } \alpha_t = \beta_t + 1,$$
$$\alpha(x) = n_1\omega^{\alpha_1} + \ldots + n_{t-1}\omega^{\alpha_{t-1}} + (n_t - 1)\omega^{\alpha_t} + \omega^{\alpha_t(x)} \qquad \text{if } \alpha \text{ is limit.}$$

A function f *dominates* g if there is a n_0 such that $f(n) > g(n)$ for all $n > n_0$. Remember that the closure of Grzegorczyk's class $\mathcal{E}_m$ under one application of the scheme of primitive recursion is $\mathcal{E}_{m+1}$ and that the Ackermann function $f_x(x)$ dominates all primitive recursive functions (see Rose [1984], Chapter 2). In general, the functions of Grzegorczyk's class $\mathcal{E}_\beta$ are dominated by f_α, with $\beta < \alpha$.

No Counterexample Interpretation. (Kreisel [1952], see also Tait [1965 a] and [1965 b]) *Let $\phi : \forall y_1 \exists x_1 \ldots \forall y_n \exists x_n . \psi(y_1, x_1, y_2, \ldots y_n, x_n)$ be a theorem of* **PA**, *with ψ quantifier-free.*[3] *Then there exist functionals X_1, ..., X_n, which are α-recursive in $\vec{\mathbf{y}} = \mathbf{y}_1, \ldots, \mathbf{y}_n$ such that if ϕ_F is*

$$\phi_F : \qquad \psi(\mathbf{y}_1, X_1(\vec{\mathbf{y}}), \mathbf{y}_2[X_1(\vec{\mathbf{y}})], \ldots, \mathbf{y}_n[X_1(\vec{\mathbf{y}}), \ldots, X_{n-1}(\vec{\mathbf{y}})], X_n(\vec{\mathbf{y}}))$$

then $\phi_F\left[\begin{smallmatrix}\vec{y}\\ \vec{f}\end{smallmatrix}\right]$ is true for every choice of (appropriate) numerical functions $\vec{f}$ for $\vec{\mathbf{y}}$. ■

We may choose the Herbrand functions using the least number principle: $(\forall y.\phi(y)) \equiv \phi(\mu y.\neg\phi(y))$. Then

Corollary. *Let ϕ be $\forall x.\exists y.\psi(x, y)$ and let F be a function(al) such that $F(a)$ is the least y such that $\psi(a, y)$ holds. If F dominates f_α for all $\alpha < \epsilon_0$, then* **PA** *cannot prove ϕ.* ■

Finally, as G. Kreisel has often pointed out, the following fact is very useful in application. To prove it, use the fact that the functional interpretation of a Π^0_1 lemma $\forall x.A(x)$ is $A(\mathbf{x})$, where $\mathbf{x}$ is 0-ary.

Proposition. *The functional* F *such that $\forall x.\phi(x, F(x)$, constructed by interpreting a given proof of $\forall x.\exists y.\phi(x, y)$, does not depend on the proofs of Π^0_1 Lemmas.* ■

[3] For expository purpose it is convenient to consider prenex formulas.

10.6. A Primitive Functional Interpretation.

Consider the induction rule in the form

$$\frac{\vdash \Gamma, A(0) \qquad \vdash \Gamma, A^{\perp}(n), A(n+1)}{\vdash \Gamma, A(t)}$$

where $A(z)$ is in Π_2 form and t is any term. Suppose the upper sequents have Herbrand's expansions $\Gamma_H^{\vec{\mathbf{w}}^0}, B(\mathbf{x}, \mathbf{s}_0, 0)$ and $\Gamma_H^{\vec{\mathbf{w}}^{n+1}}, B^{\perp}(\mathbf{t}_n, \mathbf{y}[\mathbf{t}_n], n), B(\mathbf{x}, \mathbf{s}_{n+1}, n+1)$. and let $\vec{\mathbf{v}}$ be a list of all the Herbrand functions in Γ_H. Define

$$\begin{aligned} \mathbf{f}(0, \vec{\mathbf{v}}, \mathbf{x}) &= \mathbf{s}_0; \\ \mathbf{f}(n+1, \vec{\mathbf{v}}, \mathbf{x}) &= \mathbf{s}_n \begin{bmatrix} \lambda \mathbf{x}.\mathbf{y}[\mathbf{x}] \\ \lambda \mathbf{x}.\mathbf{f}(n, \vec{\mathbf{v}}, \mathbf{x}) \end{bmatrix}. \end{aligned}$$

Let k be the length of $\vec{\mathbf{w}}^n$. In addition, for each $j \le k$ we define by simultaneous recursion terms $\mathbf{g}_j$ as follows. Let $\vec{\mathbf{g}}^n = \mathbf{g}_1(n, \vec{\mathbf{v}}, \mathbf{x}), \ldots, \mathbf{g}_k(n, \vec{\mathbf{v}}, \mathbf{x})$. Then

$$\begin{array}{lll} \mathbf{g}_j(0, \vec{\mathbf{v}}, \mathbf{x}) & = \mathbf{w}_j^0; & \\ \mathbf{g}_j(n+1, \vec{\mathbf{v}}, \mathbf{x}) & = \mathbf{g}_j(n, \vec{\mathbf{v}}, \mathbf{x}), & \text{if } \Gamma_H^{\vec{\mathbf{g}}^n} \text{ is true,} \\ & = \mathbf{w}_j^{n+1} & \text{otherwise.} \end{array}$$

Let Π_2-**Cut-IR** the system based on the language $\mathcal{L}$ of Peano Arithmetic and on the logical rules for the sequent calculus **LK** and axiom and rules for first order Peano Arithmetic, where the rule of Cut is restricted to Π_2 formulas and, in addition, the Induction rule is restricted to Π_2 induction formulas, but possibly contains arbitrary passive formulas. We can define a simple form of the No Counterexample Interpretation for Π_2-**Cut-IR**.

Primitive Functional Interpretation. *Let $\mathcal{D}$ be a derivation of $\vdash \Gamma$ in Π_2-***Cut-IR** *and let $\vec{\mathbf{y}} = \mathbf{y}_1, \ldots, \mathbf{y}_n$ be all the Herbrand functions for the universal quantifiers occurring in $\mathcal{D}$. There exist functionals $\mathbf{X} = X_1, \ldots, X_n$, which are primitive recursive in $\vec{\mathbf{y}}$ such that*

$$\Gamma_H^{\mathbf{X}} \begin{bmatrix} \vec{\mathbf{y}} \\ \vec{f} \end{bmatrix}$$

is true for every choice of (appropriate) numerical functions $\vec{f}$ for $\vec{\mathbf{y}}$. ■

Despite its restrictions, the above interpretation is enough for our application to the Infinite Ramsey Theorem.

10.7. Functionals in fragments of PA.

The *primitive functional interpretation* is related to the following classical results of the proof theory of fragments of arithmetic and weak subsystem of analysis, established by Parsons and Minc and refined by Sieg [1985] and [1988]. These results provide a general method to obtain sharp upper bounds on the complexity of the functionals.

Let $(QF - \mathbf{IA})$ be the fragment of first order arithmetic **PA** in which first order quantification is allowed, but the induction formula in the induction axiom **IA** is quantifier-free; $(QF - \mathbf{IA})$ is a conservative extension of primitive recursive arithmetic **PRA**. Let $\mathcal{F}$ be any set of functions that includes the functions of Grzegorczyk's class $\mathcal{E}_3$ and let $\mathcal{F}_k$ be the k-th class of Grzegorczyk's hierarchy, relativized to $\mathcal{F}$. Let $(QF(\mathcal{F}_k) - \mathbf{IA})$ be the subsystem of $(QF - \mathbf{IA})$ where only function symbols and defining axioms for the functions of the class $\mathcal{F}_k$ are allowed. We obtain a hierarchy of proper subsystems of $(QF - \mathbf{IA})$; indeed, the reflection principle for $(QF(\mathcal{F}_m) - \mathbf{IA})$ can be proved in $(QF_{m+2} - \mathbf{IA})$ (Sieg [1988]).

Restriction of the number of applications of $(\Pi^0_2 - \mathbf{IR})$ yields the same hierarchy. Let $(\Pi^0_2(\mathcal{F}) - \mathbf{IR})_k$ be the fragment that allows at most k applications of the Π^0_2-induction rule and, in addition, only function symbols and defining axioms for the functions of $\mathcal{F}$.

Theorem (Sieg [1988]). $(\Pi^0_2(\mathcal{F}) - \mathbf{IR})_k$ *and* $(\mathrm{QF}(\mathcal{F}_k) - \mathbf{IA})$ *prove the same sentences in the language of* $(\Pi^0_2(\mathcal{F}) - \mathbf{IR})$*; this holds also for extensions of the theories by* Π^0_1 *- sentences. In particular,* $(\Pi^0_2 - \mathbf{IR})$ *is conservative over* **PRA**.

Now the corollary of the Cut-Elimination Theorem (section 10) gives the following information.

Corollary (Sieg [1988]). *If* $(\Pi^0_2(\mathcal{F}) - \mathbf{IR})_k$ *proves a* Π^0_2*-sentence* $\forall x.\exists y.\phi(x,y)$*, then there is a function(al)* $F \in \mathcal{F}_k$*, such that* $(\mathrm{QF}(\mathcal{F}_k) - \mathbf{IA})$ *proves* $\phi(a, F(a))$.

10.8. Discussion on Implementation.

If we consider the problem of extracting computational information from given mathematical proofs in practical terms, rather than as a general metamathematical issue, then the main task is to design procedures that are simple to implement, easy to manage and perspicuous in their application.

We intend to implement the No Counterexample Interpretation (NCI), which has been efficiently applied for the purpose of unwinding in previous applications (consider, for instance, Girard [1987 a] pp.484-496, Luckhardt [1989]).

To this purpose, the most striking result of our analysis is the Herbrand Theorem for sequents of prenex formulas in **MLL**, Multiplicative Linear Logic. For such a sequent Γ the Herbrand expansion $\Gamma_H^{\vec{t}}$ is such that the expansion of each formula has length one, and is obtained by substitution of the bound variables by terms $\vec{t}$ of the "Herbrand universe". Given a proof-net with conclusions Γ and without cuts, a proof net with conclusion $\Gamma_H^{\vec{t}}$ is obtained by simple substitutions and cancellation of the lowermost quantificational links. Conversely, given a proof-net with conclusion $\Gamma_H^{\vec{t}}$, a proof-net for Γ is immediately reconstructed and its correctness verified in a straightforward manner. Traditional proofs of Herbrand's Theorem involve complicated manipulations of uninspiring notations.

The same proof extends to the system $\mathbf{LD}^+$ of Direct Logic. The result can also be extended to proof-nets containing certain forms of cut. Using the same technique, for fragments of arithmetic with the Π_2-Induction Rule we can obtain a restricted form of functional interpretation, in which the functionals are *primitive recursive in* the Herbrand's functions. Again, if the Induction Rule is restricted to induction formulas in Π_2- form, the definition and the verification of correctness of the interpretation is entirely straightforward.

Unfortunately, Herbrand's Theorem does not hold for sequents of non-prenex formulas in Multiplicative Linear Logic. Also this result cannot be extended to Additive Linear Logic. Finally, it can be easily shown that if the quantificational structure of the induction formula is more complex than a Π_2-form, a more complex definition of functional interpretation is needed.

We must emphasize that *notational transparency* is an essential advantage for the application of a procedure and plays an essential role in software design. In fact, the mechanization of a procedure can certainly take care of syntactic details that are too boring for humans; but the usefulness of mechanization is much reduced, if the procedures are very sensitive to accidental syntactic features of the formalization and if the effect of variants in the computation is not transparent in the output Proof-nets fully exhibit the structure of deductive relevance. In **MLL** the structure

of relevance is fully transparent in the conclusions. Therefore when the Herbrand terms are defined for **MLL** according to our procedure, the history and the role of each term is immediately clear from the output.

On the other hand, the method of proof-nets, and especially the verification of the *parameters condition*, may be criticized as unintuitive, when compared to sequent calculi. However, if proof-nets seem hard to read, it is because they are a particularly *concise* notation. Concision is an important advantage for large scale formalizations and justifies efforts to "adjust our intuition". Moreover, the parameters condition can be replaced by a system of $\bigwedge$-boxes together with a simple restriction on parameters (Girard [1987b]). Such a notation may be regarded as a system of *windows* and is graphically suggestive.

Thus in our application we proceed according to the following policies.

• We will formalize a proof of Infinite Ramsey Theorem (with exponent 2) in the fragment of first order Peano Arithmetic in which the Induction Rule and the Cut rule are restricted to Π_2-induction and Cut formulas.

• Fragments of the proof without the inductioon rule will be isolated and represented as derivations in **LK** (with arithmetic axioms).

• A derivation of $\vdash \Gamma$ in first order **LK** (with arithmetic axioms) will be obtained by introducing suitable application of Contraction from a derivation in first order $\mathbf{DL}^+$ of $\vdash \Gamma'$ (with arithmetic axioms), where Γ' is classically equivalent to Γ.

• Derivations in first order $\mathbf{DL}^+$ will be represented by systems of proof-nets with non-logical axioms.

• Girard's restriction on $\bigwedge$-*boxes* will be used to verify first-order correctness, instead of the *parameters condition*.

• In addition, in each proof-net (i.e., in each box) the property will hold that no propositional link may occur below any quantificational link.

• The Herbrand Theorem for prenex formulas of $\mathbf{LD}^+$ is then applied within each box.

• For each **LK** derivation a valid Herbrand expansions is defined using `if ... then ... else ...` terms.

• Finally the NCI is defined for the entire derivation, in its simplest form, the *primitive functional interpretation*.

In the documentation below we will present only

(1) the sequents formalizing the conclusions of each **LK** fragment of the derivation;

(2) the sequents formalizing the conclusions of each $\mathbf{DL}^+$ fragment;

(3) the corresponding system of proof nets;

(4) the terms resulting from application of the Herbrand Theorem;

(5) the functionals resulting for the application of the NCI.

The remaining steps of the procedure will then be obvious.

11. Proofs of Ramsey's Theorems.

In the following informal arguments we stress the structural similarity of the proofs of the Infinite Ramsey Theorem (IRT), the Finite Ramsey Theorem (FRT) and the Ramsey-Paris-Harrington Theorem (RPH). We consider only $c = k = 2$, namely, 2–colorings of graphs. It is enough to work with ordered pairs (x, y), where $x < y$. We write $r(l)$ for the Ramsey function $R(2,2,l)$ and $lr(n_1)$ for the Ramsey-Paris-Harrington function $LR(2,2,n_1)$.

11.1. Proof of the Infinite Ramsey Theorem.

As explained in Section (1.2.1), we distinguish two parts. The first part consists of three steps. Let $\chi : [\mathrm{N}]^2 \to 2$ be given.

Part 1, step 1.[4] For $i \in N$ define a chain of subsets $S(i)$ of N as follows. Let $S(-1) = \mathrm{N}$; let *Green* $(n) = \{y : \chi(n, y) = 0\}$, and *Red* $(n) = \{y : \chi(n, y) = 1\}$. Finally, let $T^0(n)$ be *Green* $(n) \cap S(n-1)$ and let $T^1(n)$ be *Red* $(n) \cap S(n-1)$.

$$S(n) = \begin{cases} S(n-1), & \text{if } n \notin S(n-1); \\ T^0(n), & \text{if } n \in S(n-1) \text{ and } |T^0(n)| \geq |T^1(n)|; \\ T^1(n) & \text{otherwise.} \end{cases}$$

By induction on n, one can prove that each $S(n)$ is unbounded.

($*$) Clearly, each $S(j)$ is a subset of $S(i)$, for all $i < j$.

Step 2. Define the "diagonal intersection" H_1 of the sets $S(n)$ by course-of-value recursion:

$$H_1 = \{x : \forall d \in H_1 \cap [x].x \in S(d)\}$$

We prove that H_1 is unbounded. Suppose b is a bound for H_1 and consider $S(b)$. For any element d of H_1, $d < b$ implies $S(b) \subset S(d)$, by ($*$). It follows that every element of $S(b)$ satisfies the definition of H_1, i.e., $S(b) \subset H_1$. Thus $S(b)$ is bounded by b, contradicting step 1.

Step 3. Define an *induced coloring* $\chi^{stab} : \mathrm{N} \to 2$ by

$$\chi^{stab}(n) = \begin{cases} 0, & \text{if } n \in S(n-1) \text{ and } T^0(n) \text{ is unbounded;} \\ 1, & \text{otherwise.} \end{cases}$$

[4] Steps 1 and 2 are usually combined.

(**) Clearly, $\chi^{stab}(n) = 0 \vee \chi^{stab}(n) = 1$, for all n.

Finally, let $U^j = \{x \in H_1 : \chi^{stab}(x) = j\}$ for $j = 0, 1$. Define

$$H = \begin{cases} U^0, & \text{if } |U^0| \geq |U^1|; \\ U^1 & \text{otherwise.} \end{cases}$$

Since H_1 is unbounded and χ^{stab} is a 2-coloring of H_1, H must be unbounded, too.

Part 2. It remains to verify that H is χ-homogeneous.

$(***)$ Clearly, if $x \in S(n)$ then $\chi(n, x) = \chi^{stab}(n)$ – by the definitions of $S(n)$ and χ^{stab}.

We need to show that there exists v such that for all $n, m \in H$ with $n < m$, $\chi(n, m) = v$; we can choose $v = 0$ if U^0 is unbounded, $v = 1$ otherwise.

Suppose U^0 is unbounded and let $n \in H$, $m \in H$ and $n < m$. It follows from the definition of H_1 that $m \in S(n)$. By definition of H, $\chi^{stab}(n) = 0$, thus $h(n, m) = 0$ by $(***)$. Since n and m are arbitrary, we are done. If U^0 is bounded, the argument is similar. ∎

11.2. Proof of the Finite Ramsey Theorem.

Let $n = 2^{2l-1} - 1$. We claim $r(l) \leq n$. Let $\chi : [n]^2 \to 2$ be given. If we take $S(-1) = [n]$ and we restrict all definitions to $[n]$, then the sets $S(i)$, H_1, χ^{stab} and H are defined as in the proof of IRT.

Steps 1 and 2 are combined. We need to show that

$$|H_1| \geq 2l - 1$$

and we prove, by induction on p, that for $1 \leq p < 2l - 1$, there is $q_p < n$ such that

$$|H_1 \cap [q_p + 1]| = p \quad \text{and} \quad |S(q_p)| \geq 2^{2l-p-1)}.$$

The computation is easy — let q_1 be $0 \in H_1$, let q_{p+1} be the least element of $S(q_p)$ (remember that $q_p < \mathit{inf}\, S(q_p)$). Finally, since $|S(q_{2l-2})| > 0$, we can pick $q_{2l-1} \leq n$. In Step 3, the *pigeon-hole principle* is invoked to conclude that, since $|H_1| \geq 2l - 1$ and χ^{stab} is a 2-coloring of H_1, we must have $|H| = l$. ∎

11.3. Proofs of the Ramsey-Paris-Harrington Theorem.

The following is an adaptation of the argument by Erdös and Mills [1981] to our context. Given n_1 we need to find n_2 such that for any $\chi : [n_1, n_2]^2 \to 2$ there is a large χ-homogeneous $H \subset [n_1, n_2]$. Let $r(n) = 2^{2n-1} - 1$. Define lr by $lr(0) = n_1 + 1$ and $lr(n+1) = lr(n) + r(lr(n))$. We claim that we can take $n_2 = lr(2n_1 - 2)$.

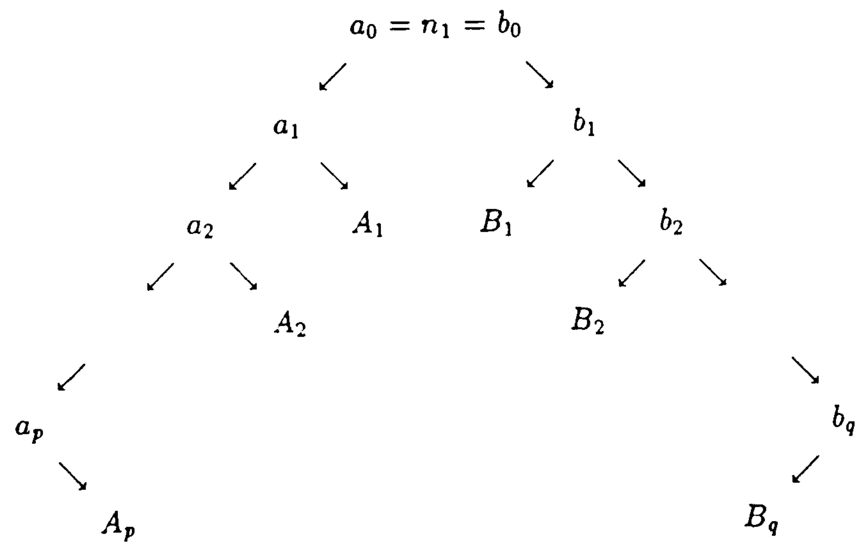
We use the following notation. Let $Green(x) = \{y : \chi(x,y) = 0\}$ and $Red(x) = \{y : \chi(x,y) = 1\}$. Let

$$\begin{aligned}
a_0 &= b_0 = n_1 \\
a_{i+1} &= \mu x . x \in \bigcap_{j \le i} Green(a_j) = \mu x . x > a_i \wedge \{a_0, \dots, a_i, x\} \text{ is } \chi\text{-homogeneous green.} \\
b_{i+1} &= \mu x . x \in \bigcap_{j \le i} Red(b_j) = \mu x . x > b_i \wedge \{b_0, \dots, b_i, x\} \text{ is } \chi\text{-homogeneous red.} \\
A_{i+1} &= \{x : a_0, \dots, a_i, x\} \text{ is } \chi\text{-homogeneous green, but } \chi(a_{i+1}, x) = 1 \text{ [= red].} \\
B_{i+1} &= \{x : b_0, \dots, b_i, x\} \text{ is } \chi\text{-homogeneous red, but } \chi(b_{i+1}, x) = 0 \text{ [= green].}
\end{aligned}$$

Finally, let $C_{i,j} = \bigcap_{l \le i} Green(a_l) \cup \bigcap_{l \le j} Red(b_l)$. Then

$$\begin{aligned}
[n_1, n_2] &= \{a_0\} \cup C_{0,0} \\
&= \{a_0, \dots, a_{i-1}\} \cup \{b_0, \dots, b_{j-1}\} \cup \bigcup_{l \le i} A_l \cup \bigcup_{l \le j} B_l \cup C_{i,j}
\end{aligned}$$

for $i \le p$, $j \le q$, $p + q \le 2n_1 - 2$. The picture is



We give two variants of the proof.

Proof 1. By induction on n, for $n \leq 2n_1 - 2$, we show that for some i, j with $i + j = n$

(i) either for some $l \leq i$, $|A_l| = r(a_l)$;

(ii) or for some $l \leq j$, $|B_l| = r(b_l)$;

(iii) or $\{a_0, \ldots, a_i\}$ is χ-homogeneous green, $\{b_0, \ldots, b_j\}$ is χ-homogeneous red and $|C_{i,j}| > n_2 - lr(n)$.

Clearly, if (i), then either there is a χ-homogeneous red set $D \subset A_l$ of cardinality a_l, and $\{a_l\} \cup D$ is χ-homogeneous and large, or there is a χ-homogeneous green set $D \subset A_l$ of cardinality a_l, and $\{a_0, \ldots, a_{l-1}\} \cup D$ is χ-homogeneous and certainly large. Similarly, if (ii). Finally, if (iii) with $i + j = 2n_1 - 2$, then $|C_{i,j}| > 0$ and either $\{a_0, \ldots, a_i, c\}$ or $\{b_0, \ldots, b_j, c\}$ is χ-homogeneous and large, where c is the least element of $C_{i,j}$.

Proof 2. Let $S(n_1) = [n_1, n_2]$. For $n \geq n_1$, we define sets $S(n)$ as follows

$$S(n) = \begin{cases} S(n-1), & \text{if } n \notin S(n-1); \\ \mathit{Green}(n) \cap S(n-1), & \text{if } n \in S(n-1) \text{ and} \\ & \quad \text{either } p = n_1; \\ & \quad \text{or } p < n_1, \text{ but } \exists j. |B_j| \geq b_j \wedge n = b_j; \\ & \quad \text{or } p < n_1, \text{ but } \exists j. |B_j| \geq b_j \wedge n > b_j \wedge \\ & \qquad \wedge |\mathit{Green}(n) \cap S(n-1)| \geq |\mathit{Red}(n) \cap S(n-1)|; \\ & \quad \text{or } q < n_1, \text{ but } \exists j. |A_j| \geq a_j \wedge n > a_j \wedge \\ & \qquad \wedge |\mathit{Green}(n) \cap S(n-1)| \geq |\mathit{Red}(n) \cap S(n-1)|; \\ \mathit{Red}(n) \cap S(n-1) & \text{otherwise.} \end{cases}$$

Now we define the "diagonal intersection" H_1 as above and show:

(i) either $p = n_1$ and $|H_1| \geq n_1$;

(ii) or $q = n_1$ and $|H_1| \geq n_1$;

(iii) or $p, q < n_1$ and $|H_1| \geq l + 2c_l$, where $c_l = a_l$ or b_l, $l < n_1$.

This replaces steps (1) and (2) in the proof of FRT.

Step (3). We define $\chi^{stab}(n)$ in accordance with the new definition of $S(n)$. We define H as above. Then, arguing as in Proof 1, we show that $|H_1| = |H| \geq n_1$ if (i) or (2); if (iii), we show that $|H| \geq n_1$ if $n_1 \in H$ and that $|H| \geq c_l$ if $c_l \in H$. We verify that

$(***)$ if $x \in S(n)$ then $\chi(n, x) = \chi^{stab}(n)$

with the new definitions of $S(n)$ and χ^{stab}. We prove that H is χ-homogeneous as before. ■

11.4. Remarks and Questions.

Remarks. (i) The RPH Theorem, as formulated in section (1.2.1), can be formulated in $\mathcal{L}_{\mathbf{PA}}$, but is not provable in **PA**. The proof in Ketonen and Solovay [1981] (see also Graham et al. [1980]) is a remarkable application of the Corollary in Section (10), since it shows that $LR(c,k,n)$ dominates every f_α. For any fixed k, however, the result *is* provable in **PA**.

(ii) Let $k=2$. Erdös and Mills [1981] show that $LR(n,2,n)$ is the Ackermann function $f_n(n)$. In particular, a coloring $\chi : [n, f_c(n)-1]^2 \to c$ is exhibited such that every χ-homogeneous set has cardinality less than n. Namely, $\chi(x,y)$ is defined to be the least $i \leq c$ such that for some j

$$x, y \in [f_i^{(j)}(n), f_i^{(j+1)}(n) - 1]$$

(see also Graham et al. [1980], p.151). For a fixed c, however, $LR(c,2,n)$ is primitive recursive.

(iii) The FRT and the RPH Theorem follow by a compactness argument from the IRT - see section (14). Since we consider only countable colorings, (Weak) König's Lemma suffices.

(iv) From (i) and (iii) it follows that the IRT is not provable in a conservative extension of **PA**. For fixed k, the IRT is provable in a conservative extension of **PA**.

Notice that the compactness argument asserts the existence of an n, for which FRT holds, but does not explicitly give it as a function of c, k and l.

Question 1: *Can we actually obtain a bound for the function $R(c,k,l)$ by the functional interpretation of the compactness argument for the Finite Ramsey Theorem from the infinite version?*

Question 2: *The above proofs of the Infinite, the Finite and the Ramsey-Paris-Harrington theorems have different conclusions and computational complexities, but similar structure. Can we formulate the common features in the form of a more general lemma?*

12. Analysis of a First Order Proof of the IRT.

We postpone consideration of the compactness argument and of *Question 1* to the section (14). We focus on a formal proof of the Infinite Ramsey Theorem, and we restrict ourselves to the proof of IRT, exponent 2. Moreover, only Part 1 of the proof in section (11.1) is computationally significant, namely, the part in which we prove that the set H is unbounded.

Some parts of the formal proof are given in great detail in sections (12.1)-(12.2) as examples of the techniques presented in Part I of the dissertation. The reader interested only in the mathematical results related to Questions 1 and 2 may omit such details and go directly to the section (13).

In particular, sections (12.1.1) and (13.3) and the Herbrand's analysis in section (13.3) may be regarded as description of a computer-aided interactive procedure of formalization and transformation of proofs. The implementation of such a procedure in a proof-checker of the type EKL is within reach of the present software technology.

The language is that of first order **PA** with additional predicate and function symbols – namely χ, χ^{stab}, S, H_1, H. The following table may be useful in comparing the proof of section (11.1) with its formalization.

The expression	is formalized as
"$x \in S(n)$"	$S(n,x)$
"$\chi : [\mathrm{N}]^2 \to 2$"	$\forall xy.[\chi(x,y) = 0 \vee \chi(x,y) = 1]$
"$\chi^{stab} : [\mathrm{N}] \to 2$"	$\forall x.[\chi^{stab}(x) = 0 \vee \chi^{stab}(x) = 1]$
"*Green* (n)"	$\chi(n,x) = 0$
"*Red* (n)"	$\chi(n,x) = 1$
"*Green* (0) *is bounded*"	$\exists b.\forall m.[\chi(0,m) = 0 \supset m < b]$
"*Green* (0) *is unbounded*"	$\forall e.\exists d.[\chi(0,d) = 0 \wedge d \geq e]$.

The endsequent of our derivation is informally expressible as

"$\chi : [\mathrm{N}]^2 \to 2$, *Definitions of* S, χ^{stab}, H_1 *and* H

$\vdash$

H *is unbounded and for some* $l < 2$, *for all distinct* $n, m \in H$, $\chi(n,m) = l$."

We divide our proof in two parts:

Part I (*Unboundedness*)

"$\chi : [\mathrm{N}]^2 \to 2$, *Definitions of* S, χ^{stab}, H_1 *and* $H \vdash$ H *is unbounded.*"

and

Part II (*Homogeneity*)

"$\chi : [\mathrm{N}]^2 \to 2$, *Definitions of* S, χ^{stab}, H_1, *and* H

$\vdash$

for some $l < 2$ and for all distinct $n, m \in H$, $\chi(n,m) = l$."

The proof of Part I is divided into three steps.

The first step yields the *unboundedness of the stable sets* $S(n)$. We write a formal derivation in Σ^0_3-**IR** [5] of

Part I, Step 1

$$\forall n.\mathit{def}\, S(n),\ \forall nq.\chi(n,q) = 0 \vee \chi(n,q) = 1 \quad \vdash \quad \forall n\ k.\exists i.S(n,i) \wedge i \geq k$$

Here *def* $S(n)$ is the formal definition of the sets $S(n)$:

$$\forall x.S(0,x) \equiv \text{“}x \in \mathrm{Green}(0) \wedge \mathrm{Green}(0) \text{ is unbounded”} \vee$$
$$\text{“}x \in \mathrm{Red}(0) \wedge \mathrm{Green}(0) \text{ is bounded”};$$

$$\forall x.S(n+1,x) \equiv [\neg S(n,n+1) \wedge S(n,x)] \vee$$
$$[S(n,n+1) \wedge \text{“}x \in \mathrm{Green}(n+1) \cap S(n) \wedge$$
$$\wedge\, \mathrm{Green}(n+1) \cap S(n) \text{ is unbounded”}] \vee$$
$$[S(n,n+1) \wedge \text{“}x \in \mathrm{Red}(n+1) \cap S(n) \wedge$$
$$\wedge\, \mathrm{Green}(n+1) \cap S(n) \text{ is bounded.”}]$$

[5] Π^0_2-**IR** is not enough here, because the side formula *def* $S(n)$ in the antecedent increases the logical complexity of the *sequent* (see W.Sieg [1985], p.40, footnote 6).

Since in the first part of the proof only one direction of the equivalence is used, we may let *def* $S(n)$ be

$$\forall x.[(\forall e.\exists d.\chi(0,d)=0 \wedge d \geq e) \wedge \chi(0,x)=0] \vee$$
$$\vee\ [(\exists b.\forall m.\chi(0,m)=0 \supset m < b) \wedge \chi(0,x)=1] \supset S(0,x),$$
$$\forall x.[\neg S(n,n+1) \wedge S(n,x)] \vee$$
$$\vee\ [S(n,n+1) \wedge (\forall e.\exists d.S(n,d) \wedge \chi(n+1,d)=0 \wedge d \geq e) \wedge$$
$$\wedge\ S(n,x) \wedge \chi(n+1,x)=0]$$
$$\vee\ [S(n,n+1) \wedge (\exists b.\forall m.S(n,m) \wedge \chi(n+1,m)=0 \supset m < b) \wedge$$
$$\wedge\ S(n,x) \wedge \chi(n+1,x)=1] \supset S(n+1,x).$$

The second step yields the *unboundedness of the diagonal set* H_1. We formalize the argument of section (11.1) to provide a derivation [6] of

Part I, Step 2

$$def\ S,\ def\ H_1,\ \forall k.\exists i.S(k,i) \wedge i \geq k \quad \vdash \quad \forall b.\exists c.H_1(c) \wedge c \geq b$$

def H_1 is a course-of-value inductive definition of the set H_1: $\forall x.H_1(x) \equiv (\forall d.d < x \wedge H_1(d) \supset S(d,x))$. In part 1 we use only the direction

$$\forall x.\big[\forall d.d < x \wedge H_1(d) \supset S(d,x)\big] \supset H_1(x).$$

The third step is the proof of *unboundedness of the homogeneous set* H. We write a formal derivation [13] of

Part I, Step 3

$$def\ H,\ \forall n.\chi^{stab}(n)=0 \vee \chi^{stab}(n)=1,\ \forall b.\exists c.H_1(c) \wedge c \geq b$$
$$\vdash$$
$$\forall j.\exists m.H(m) \wedge m \geq j$$

[6] In Steps 2 and 3 induction is only used to prove the facts (*) and (**). These are Π^0_1 lemmas, thus they are irrelevant for the computation of the bounds, according to the Proposition of section (10). However, by the previous footnote, these inductions are formally instances of the $\Sigma^0_3 -$ **IR**.

def H is the explicit definition of the set H:

$$\forall x. H(x) \equiv [\text{“}x \in U^0 \wedge U^0 \text{ is unbounded”}] \vee [\text{“}x \in U^1 \wedge U^0 \text{ is bounded”}]$$

In part 1 we use only the direction

$$\begin{aligned}\forall x.\ &[(\forall p.\exists q.H_1(q) \wedge \chi^{stab}(q) = 0 \wedge q \geq p) \wedge H_1(x) \wedge \chi^{stab}(x)] \vee \\ &\vee [(\exists r.\forall u.H_1(u) \wedge \chi^{stab}(u) = 0 \supset u < r) \wedge H_1(x) \wedge \chi^{stab}(x)] \supset H(x)\end{aligned}$$

The only property of χ^{stab} relevant here is $\chi^{stab} : [\mathrm{N}] \to 2$.

The proof of Part I (*Unboundedness*) is concluded as follows: first, Cut is applied to *Step 1* and **Step 2** with Cut-formula

$$\forall k.\exists i.S(k,i) \wedge i \geq k$$

then another Cut is applied to the resulting sequent and **Step 3**, with Cut-formula

$$\forall b.\exists c.H_1(c) \wedge c \geq b.$$

12.1. A Preliminary Analysis of Step 1.

We begin the construction of a formal derivation of Part I, (*Unboundedness*). We consider a possible formalization of the proof of Step 1, the unboundedness of the sets $S(n)$. Write

Step 1_0

$$\forall q.\mathit{def}\ \ S(q),\ \text{“}\chi : [\mathrm{N}]^2 \to 2\text{”} \quad \vdash \quad \forall k.\exists i.S(0,i) \wedge i > k$$

for the base case, and

Step 1^n_{n+1}

$$\forall q.\mathit{def}\ \ S(q),\ \text{“}\chi : [\mathrm{N}]^2 \to 2\text{”}, \forall c.\exists a.S(n,a) \wedge a > c \vdash \forall k.\exists i.S(n+1,i) \wedge i > k$$

for the induction step. An informal argument for the base case is as follows.

Suppose $S(0)$ to be bounded, in order to find a contradiction. Let k be such that $maxS(0) < k$.

Part (1). Assume *Green*(0) is unbounded. Let m be any element of *Green*(0). By the definition of $S(0)$ and the assumption of part (1) (*first use of the assumption*), $m \in S(0)$ and therefore $m < k$, by the assumption that k is a bound for $S(0)$. Since m is arbitrary, we infer that *Green*(0) is bounded, contradicting the assumption (*second use*). We conclude that *Green*(0) is bounded.

Part (2). Assume *Green*(0) is bounded and let e be such a bound (*first use of the assumption*). Let t be $max(e, k)$. If $\chi(0, t) = 0$, then $t \in Green(0)$ and so $t < e$, a contradiction. If $\chi(0, t) = 1$, then by the definition of $S(0)$ and by the assumption of part (2) (*second use*) t belongs to $S(0)$ and therefore we obtain $t < k$, again a contradiction. Since in both cases we obtain a contradiction, we conclude that the set $S(0)$ itself is unbounded.

We conclude the proof by taking the conclusion of part (1) for the assumption of part (2) (*Cut*). ■

Remarks. (i) The proof of the induction step **Step** $\mathbf{1}^{n}_{n+1}$ is similar. Part (1) is the same, with the replacement of $Green(n+1) \cap S(n)$ for *Green*(0). Part (2) is as follows:

(2) Assume $Green(n+1) \cap S(n)$ is bounded and let e be such a bound. Let t be $max(e, k)$. By the induction hypothesis $S(n)$ is unbounded, thus there is an $a \in S(n)$ such that $a \geq t$. If $\chi(0, a) = 0$, then $a \in Green(n+1) \cap S(n)$ and so $t < e$, a contradiction. If $\chi(0, a) = 1$, then by the definition of $S(n+1)$ and by the assumption of Part (2) we obtain that a belongs to $S(n+1)$ and therefore $a < k$, again a contradiction. Since in both cases we obtain a contradiction, we conclude that the set $S(n+1)$ itself is unbounded.

The proof that H is unbounded (**Step 3**) has the same structure as the proof of **Step** $\mathbf{1}^{n}_{n+1}$.

(ii) The definition of $S(0)$ and of χ^{stab} are of the form *if A then B else C*. It seems that an appropriate formalization of conditional expressions in Linear Logic should use *additive* conjunction, e.g., it should have the form $(A \supset B) \sqcap (\neg A \supset C)$. We do not pursue such an idea, since the use of Direct Logic (*multiplicative* interpretation) is much more convenient for the purpose of the Herbrand Theorem and of the No Counterexample Interpretation.

(iii) The Cut indicated at the end of our informal argument has the property that in both parts (1) and (2) the Cut formula "*Green* (0) *is bounded*" results from two uses of an assumption. Therefore this is an example of a Cut of the form

$$
\begin{array}{llll}
S_1: & \underline{\Gamma \vdash \Delta, A, A} & S_3: & \underline{A, A, \Pi \vdash \Lambda} \\
S_2: & \Gamma \vdash \Delta, A & S_4: & A, \Pi \vdash \Lambda \\
\cline{2-4}
S_5: & & \Gamma, \Pi \vdash \Delta, \Lambda & .
\end{array}
$$

discussed in section (1.1.1).

Let $t = \max(p, q)$.

Here $t \geq p$ and $t \geq q$ are arithmetic axioms.

Eigenvariables: $q <^0 p$

Herbrand's functions: $q \rightsquigarrow q \qquad p \rightsquigarrow p$

$$\dfrac{\neg h(0,t) = 0 \qquad \neg h(0,t) = 1}{\dfrac{\neg[h(0,t) = 0 \lor h(0,t) = 1]}{\neg\forall m.[h(0,m) = 0 \lor h(0,m) = 1]}}$$

$$\dfrac{h(0,t) = 0 \quad t \geq p}{h(0,t) = 0 \land t \geq p} \qquad \dfrac{h(0,t) = 1 \quad t \geq q}{h(0,t) = 1 \land t \geq q}$$

$$[h(0,t) = 0 \land t \geq p] \lor [h(0,t) = 1 \land t \geq q]$$

$$\exists j.[h(0,i) = 0 \land i \geq p] \lor [h(0,t) = 1 \land t \geq q]$$

$$\exists ij.[[h(0,i) = 0 \land i \geq p] \lor [h(0,j) = 1 \land j \geq q]]$$

$$\neg\forall m.[h(0,m) = 0 \lor h(0,m) = 1] \qquad \forall pq.\exists ij.[[h(0,i) = 0 \land i \geq p] \lor [h(0,j) = 1 \land j \geq q]]$$

Figure 1.

Let $t = \max(p, q)$ and let n' be $n + 1$. Aritmetic Axioms: $a \geq p \vee \neg a \geq t$ and $a \geq q \vee \neg a \geq t$.

Eigenvariables: $a <^0 q <^0 p$ .Herbrand's functions: $a \rightsquigarrow a[t]$ $q \rightsquigarrow q$ $p \rightsquigarrow p$

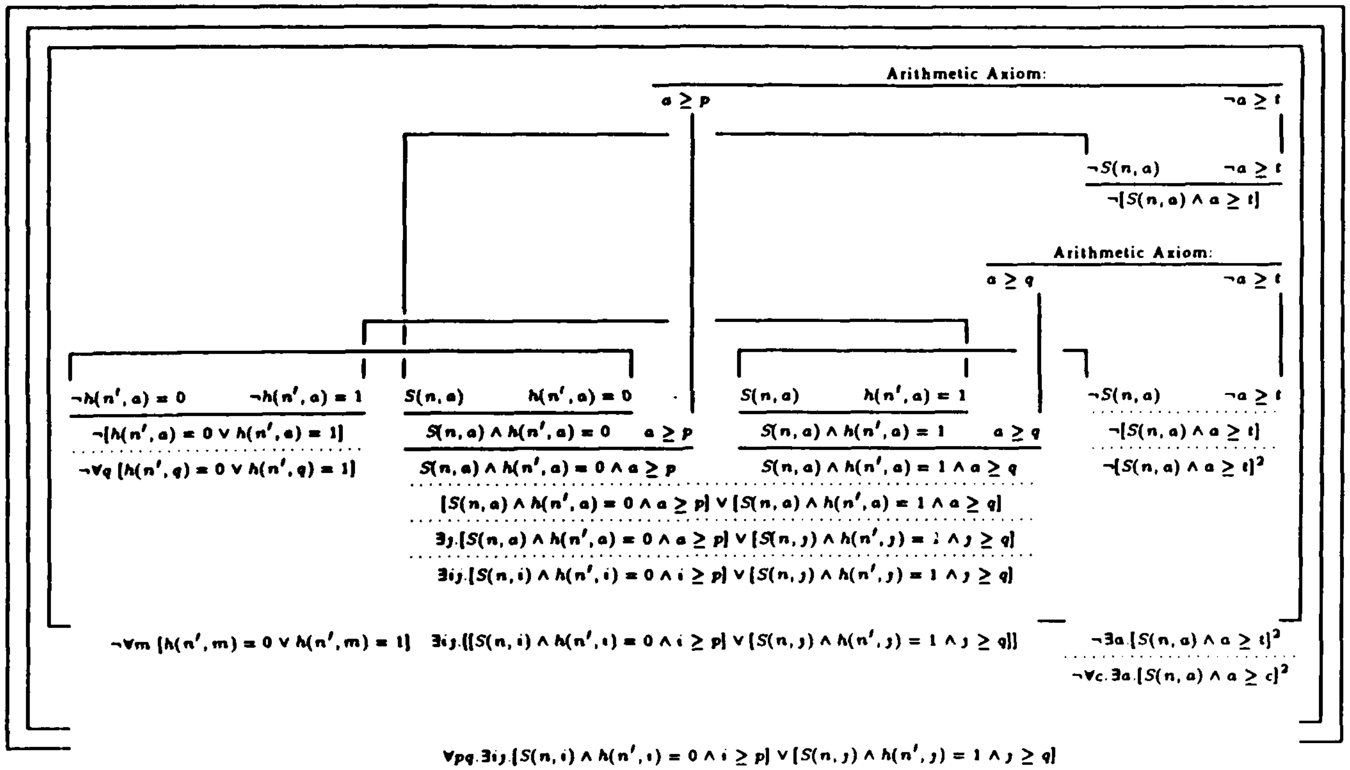
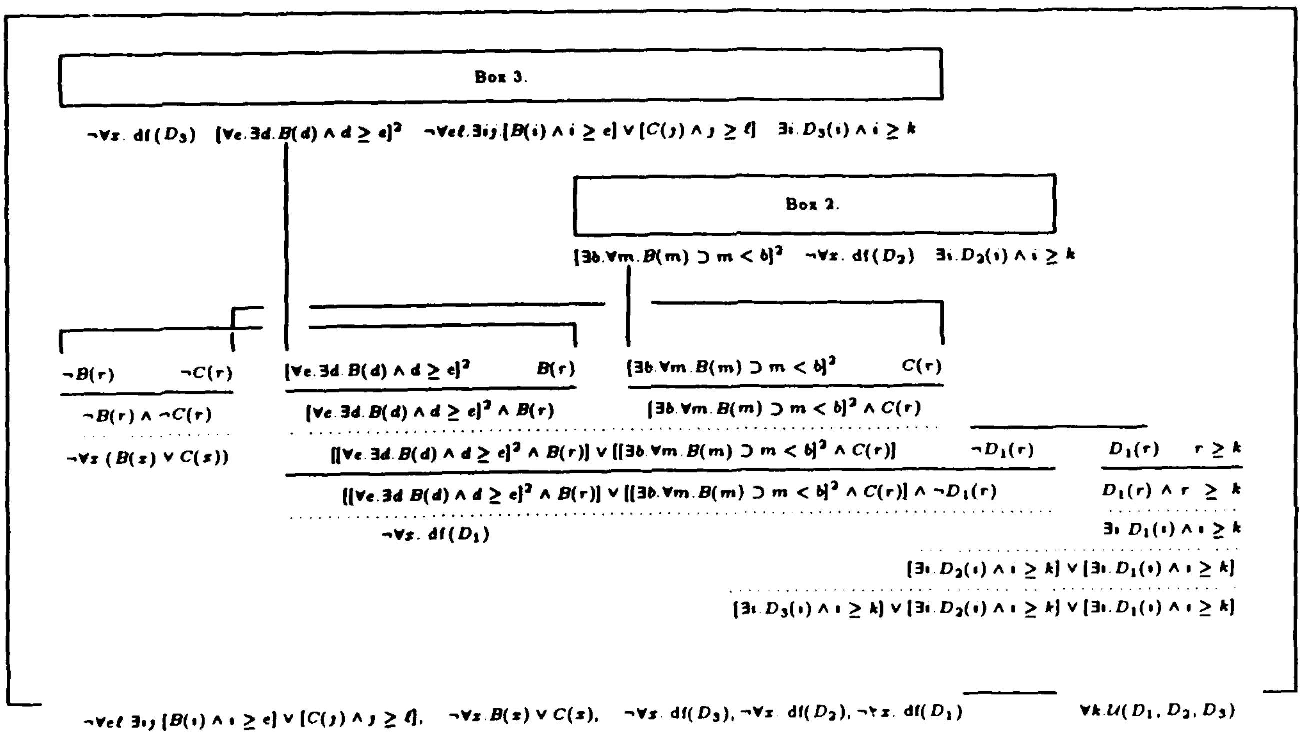


Figure 2.

Let $df(D_1)$ be $[[\forall e.\exists d.B(d) \wedge d \geq e]^2 \wedge B(x)] \vee [[\exists b.\forall m.B(m) \supset m < b]^2 \wedge C(x)] \supset D_1(x)$,

let $U(D_1, D_2, D_3)$ be $(\exists i.D_2(i) \wedge i \geq k) \vee (\exists i.D_3(i) \wedge i \geq k) \vee (\exists i.D_1(i) \wedge i \geq k)$;

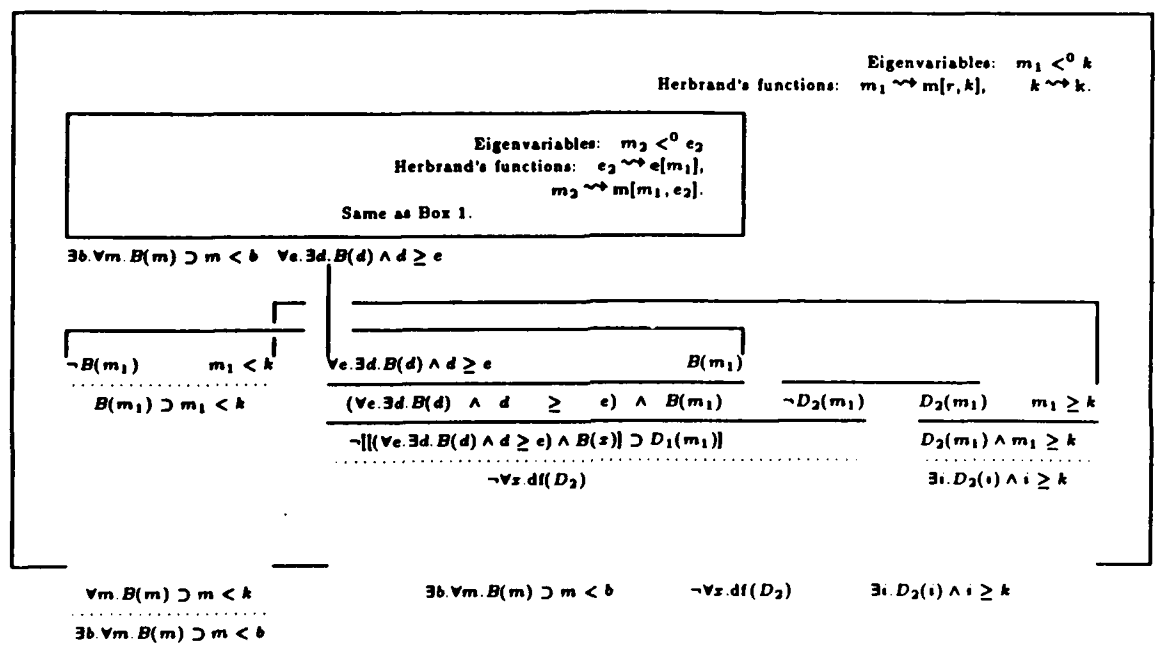
and fix $r \geq k$.



Box 4

Figure 3.

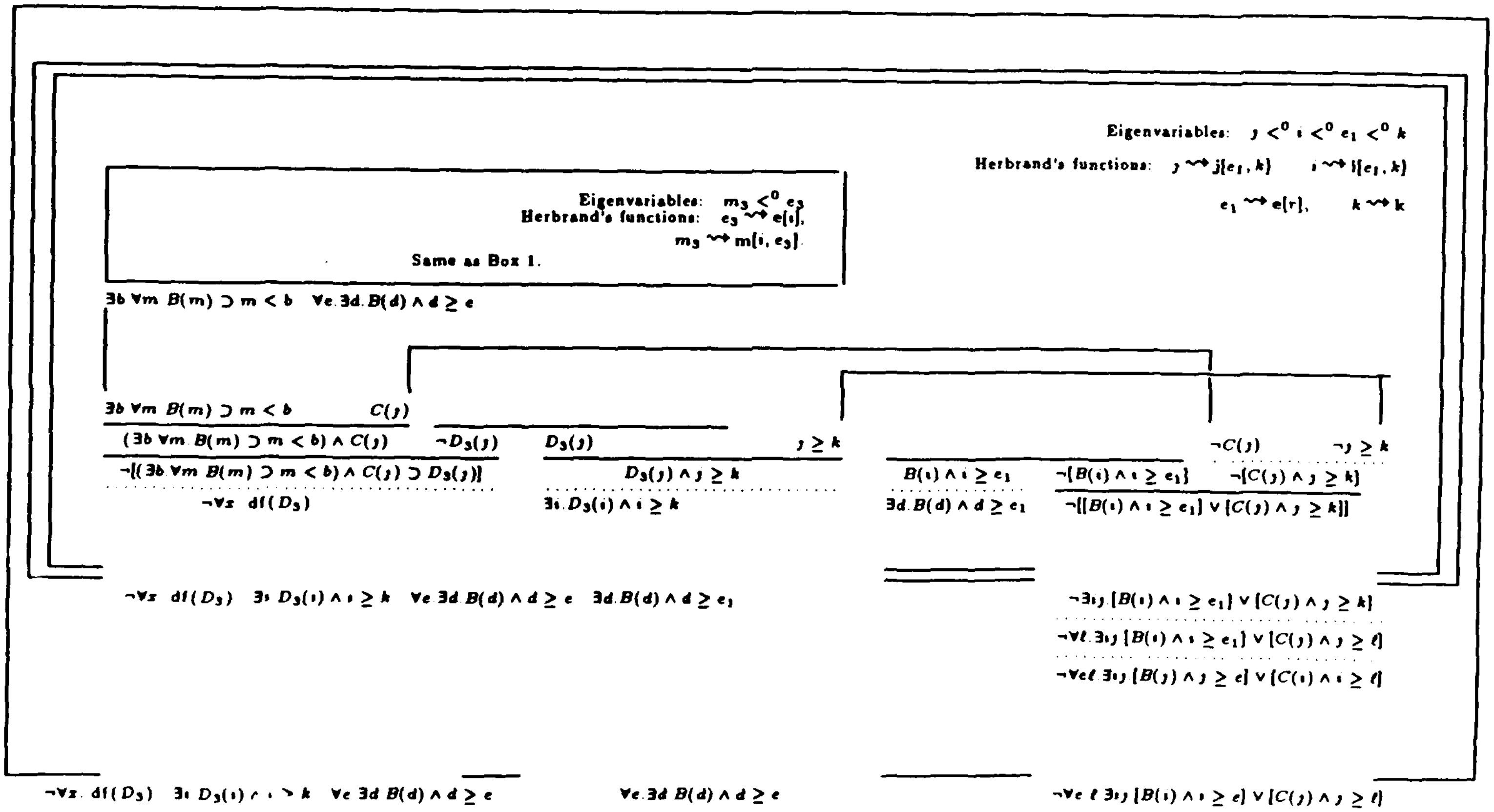
Let df(D_2) be $[[\forall e.\exists d.B(d) \wedge d \geq e] \wedge B(s)] \supset D_2(s)$



Box 2.

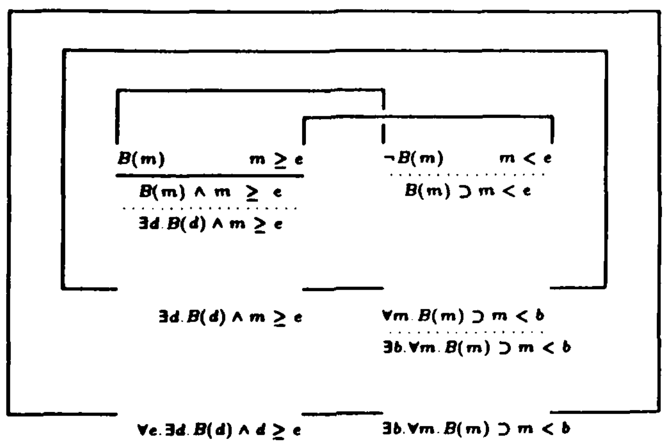
Figure 4.

Let $df(D_3)$ be $\neg[(\exists b.\forall m.B(m) \supset m < b) \wedge C(x) \supset D_3(x)]$



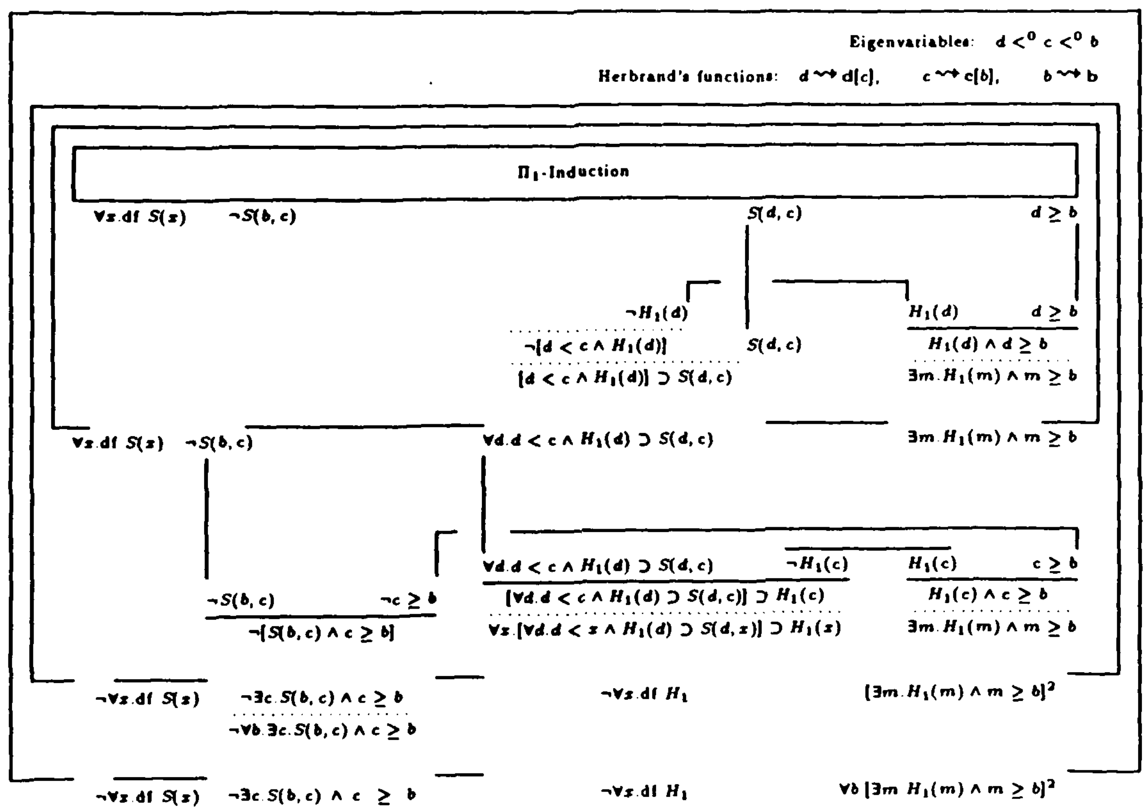
Box 3

Figure 5



Box 1.

Figure 6.



Eigenvariables: $d <^0 c <^0 b$
Herbrand's functions: $d \rightsquigarrow d[c]$, $c \rightsquigarrow c[b]$, $b \rightsquigarrow b$
Π_1-Induction
$\forall x.\mathrm{df}\ S(x)$ $\neg S(b,c)$ $S(d,c)$ $d \geq b$
$\neg H_1(d)$ $H_1(d)$ $d \geq b$
$\neg[d < c \wedge H_1(d)]$ $S(d,c)$ $H_1(d) \wedge d \geq b$
$[d < c \wedge H_1(d)] \supset S(d,c)$ $\exists m.H_1(m) \wedge m \geq b$
$\forall x.\mathrm{df}\ S(x)$ $\neg S(b,c)$ $\forall d.d < c \wedge H_1(d) \supset S(d,c)$ $\exists m.H_1(m) \wedge m \geq b$
$\forall d.d < c \wedge H_1(d) \supset S(d,c)$ $\neg H_1(c)$ $H_1(c)$ $c \geq b$
$\neg S(b,c)$ $\neg c \geq b$ $[\forall d.d < c \wedge H_1(d) \supset S(d,c)] \supset H_1(c)$ $H_1(c) \wedge c \geq b$
$\neg[S(b,c) \wedge c \geq b]$ $\forall x.[\forall d.d < x \wedge H_1(d) \supset S(d,x)] \supset H_1(x)$ $\exists m.H_1(m) \wedge m \geq b$
$\neg\forall x.\mathrm{df}\ S(x)$ $\neg\exists c.S(b,c) \wedge c \geq b$ $\neg\forall x.\mathrm{df}\ H_1$ $[\exists m.H_1(m) \wedge m \geq b]^2$
$\neg\forall b.\exists c.S(b,c) \wedge c \geq b$
$\neg\forall x.\mathrm{df}\ S(x)$ $\neg\exists c.S(b,c) \wedge c \geq b$ $\neg\forall x.\mathrm{df}\ H_1$ $\forall b\,[\exists m\ H_1(m) \wedge m \geq b]^2$

Figure 7.

Let $t = \max(p, q)$.

Arithmetic Axiom: $a \geq p \vee a \geq q \vee \neg a \geq t$.

Eigenvariables: $a <^0 q <^0 p$

Herbrand's functions: $a \rightsquigarrow a[t]$ $\quad q \rightsquigarrow q \quad p \rightsquigarrow p$

Arithmetic Axiom:

$a \geq p \qquad a \geq q \qquad \neg a \geq t$

$$\dfrac{\neg\chi^{stab}(a) = 0 \qquad \neg\chi^{stab}(a) = 1}{\dfrac{\neg[\chi^{stab}(a) = 0 \vee \chi^{stab}(a) = 1]}{\neg\forall m\,[\chi^{stab}(m) = 0 \vee \chi^{stab}(m) = 1]}}$$

$$\dfrac{\dfrac{H_1(a) \qquad \chi^{stab}(a) = 0}{H_1(a) \wedge \chi^{stab}(a) = 0} \qquad a \geq p}{H_1(a) \wedge \chi^{stab}(a) = 0 \wedge a \geq p}$$

$$\dfrac{\dfrac{H_1(a) \qquad \chi^{stab}(a) = 1}{H_1(a) \wedge \chi^{stab}(a) = 1} \qquad a \geq q}{H_1(a) \wedge \chi^{stab}(a) = 1 \wedge a \geq q}$$

$$[H_1(a) \wedge \chi^{stab}(a) = 0 \wedge a \geq p] \vee [H_1(a) \wedge \chi^{stab}(a) = 1 \wedge a \geq q]$$

$$\exists j\,[H_1(a) \wedge \chi^{stab}(a) = 0 \wedge a \geq p] \vee [H_1(j) \wedge \chi^{stab}(j) = 1 \wedge j \geq q]$$

$$\exists ij\,[H_1(i) \wedge \chi^{stab}(i) = 0 \wedge i \geq p] \vee [H_1(j) \wedge \chi^{stab}(j) = 1 \wedge j \geq q]$$

$$\dfrac{\dfrac{\neg H_1(a) \qquad \neg H_1(a)}{\neg[H_1(a) \wedge H_1(a)]} \qquad \neg a \geq t}{\neg[H_1(a) \wedge H_1(a) \wedge a \geq t]}$$

$$\neg\forall m\,[\chi^{stab}(m) = 0 \vee \chi^{stab}(m) = 1] \quad \exists ij\,[[H_1(i) \wedge \chi^{stab}(i) = 0 \wedge i \geq p] \vee [H_1(j) \wedge \chi^{stab}(j) = 1 \wedge j \geq q]]$$

$$\dfrac{\neg\exists a\,[H_1(a) \wedge H_1(a) \wedge a \geq t]}{\neg\forall c\,\exists a\,[H_1(a) \wedge H_1(a) \wedge a \geq c]}$$

$$\forall pq\,\exists ij\,[H_1(i) \wedge \chi^{stab}(i) = 0 \wedge i \geq p] \vee [H_1(j) \wedge \chi^{stab}(j) = 1 \wedge j \geq q]$$

Figure 8

12.1.1. Analysis of Steps 1 and 3.

We shall give a different formal analysis of the arguments for **Step 1** and for **Step 3**. We produce derivations of sequents classically equivalent to **Step** $\mathbf{1}_0$, **Step** $\mathbf{1}^n_{n+1}$ and to **Step 3**, in Direct Logic with Π_2 Cut. More precisely, each of the proofs of **Step** $\mathbf{1}_0$, **Step** $\mathbf{1}^n_{n+1}$ and of **Step 3** can be analyzed in two subarguments[7]: the first subargument can be carried through in Linear Logic. The second subargument cannot be formalized in Direct Logic, but a derivation exists if the conclusion is expanded by iteration of some subformulas. Finally, the two parts are connected by Cut, with a Π_2-Cut-formula. In conclusion, the Herbrand Theorem for Direct Logic with Π_2-Cut can be applied to the resulting derivation.

The first subargument for the base case is a proof of

"*if* $\chi : [\mathrm{N}]^2 \to 2$, *then either Green* (0) *or Red* (0) *is unbounded*", i.e.,

Step $\mathbf{0}_0$

$$\forall n\, m.\chi(n,m) = 0 \vee \chi(n,m) = 1 \quad \vdash$$
$$\forall p\, q.\exists i\, j.(\chi(0,j) = 0 \wedge j \geq p) \vee (\chi(0,i) = 0 \wedge i \geq q)$$

The argument for **Step** $\mathbf{0}_0$ is as follows.

Suppose for all m, $\chi(0,m) = 0$ or $\chi(0,m) = 1$. Let p and q be arbitrary, let $t = \max(p,q)$ and take $i = t = j$. Thus either $\chi(0,t) = 0$ and $t \geq p$ or $\chi(0,t) = 1$ and $t \geq q$, and we are done. ∎

A proof-net corresponding to a derivation of **Step** $\mathbf{0}_0$ is given in Figure 1.

[7] I found this organization of the argument in J.Ketonen's proof, mechanized with EKL.

The first subargument for the induction step is a proof of "*if* $\chi : [\mathrm{N}]^2 \rightarrow 2$ *and* $S(n)$ *is unbounded, then either Green* $(n+1) \cap S(n)$ *or Red* $(n+1) \cap S(n)$ *is unbounded*", i.e.,

Step $\mathbf{0}^n_{n+1}$

$$\forall n\ q.\chi(n,q) = 0 \vee \chi(n,q) = 1, \forall c.\exists a.S(n,a) \wedge a \geq c,$$
$$\vdash$$
$$\forall p\ q.\exists i\ j.[S(n,j) \wedge \chi(n+1,j) \wedge j \geq p] \vee [S(n,i) \wedge \chi(n+1,i) \wedge i \geq q];$$

The argument for **Step $\mathbf{0}^n_{n+1}$** is as follows.

Suppose for all m, $\chi(n+1,m) = 0$ or $\chi(n+1,m) = 1$. The induction hypothesis is that $S(n)$ is unbounded. Let p and q be arbitrary and let $t = \max(p,q)$. By the induction hypothesis there is an $a \in S(n)$ such that $a \geq t$. Thus either $\chi(0,a) = 0$ and $a \geq e$, or $\chi(0,a) = 1$ and $a \geq \ell$, and we are done. ■

A proof-net corresponding to a derivation of **Step $\mathbf{0}^n_{n+1}$** is given in Figure 2.

The first subargument for **Step 3** is a proof of "*if* $\chi^{stab} : [H_1] \rightarrow 2$, H_1 *is unbounded, then either* $\{z \in H_1 : \chi^{stab}(z) = 0\}$ *or* $\{z \in H_1 : \chi^{stab}(z) = 1\}$ *is unbounded*", i.e.,

Step $\mathbf{3}_0$

$$\forall n.\chi^{stab}(n) = 0 \vee \chi^{stab}(n) = 1, \forall b.\exists c.H_1(c) \wedge c \geq b,$$
$$\vdash$$
$$\forall p\ q.\exists i\ j.[H_1(j) \wedge \chi^{stab}(j) = 0 \wedge j \geq p] \vee [H_1(i) \wedge \chi^{stab}(i) = 1 \wedge i \geq q);$$

The argument for **Step $\mathbf{3}_0$** has the same structure as that for **Step $\mathbf{0}^n_{n+1}$**. A proof-net corresponding to a derivation of **Step $\mathbf{3}_0$** is given in Figure 8.

The abstract form of second part is the following Step (**00**):

Step (00)

$$\forall e\ \ell.\exists ij.[B(i) \wedge i \geq e] \vee [C(j) \wedge j \geq \ell], \quad \forall x.B(x) \vee C(x),$$

$$\forall x.[[\forall e.\exists d.B(d) \wedge d \geq e] \wedge B(x)] \vee [[\exists b.\forall m.B(m) \supset m < b] \wedge C(x)] \supset D(x),$$

$$\vdash \quad \forall k.\exists i.D(i) \wedge i \geq k$$

To obtain **Step** $\mathbf{1}_0$ we replace $\chi(0,x) = 0$ for $B(x)$, $\chi(0,x) = 1$ for $C(x)$ and $S(0,x)$ for $D(x)$ in **Step (00)**.

To obtain **Step** $\mathbf{1}^n_{n+1}$ we replace $S(n,x) \wedge \chi(n+1,x) = 0$ for $B(x)$, $S(n,x) \wedge \chi(n+1,x) = 1$ for $C(x)$ and $S(n+1,x)$ for $D(x)$. [8]

Finally, to obtain **Step 3** we replace $H_1(x) \wedge \chi^{stab}(x) = 0$ for $B(x)$, $H_1(x) \wedge \chi^{stab}(x) = 1$ for $C(x)$ and $H(x)$ for $D(x)$.

[8] More exactly, one needs

$$\begin{gathered}
\forall e\ \ell.\exists i\ j.(B(i) \wedge i \geq e) \vee (C(j) \wedge j \geq \ell), \\
\forall x.[[\neg S(n,n+1) \wedge S(n,x)] \vee \\
\vee [S(n,n+1) \wedge (\forall e.\exists d.B(d) \wedge d \geq e) \wedge B(x)] \vee \\
\vee [S(n,n+1) \wedge (\exists b.\forall m.B(m) \supset m < b) \wedge C(x)]] \supset D(x) \\
\vdash \\
\forall k.\exists i.D(i) \wedge i \geq k
\end{gathered}$$

We work with the simpler sequent. Notice that the two sequents have Herbrand terms of the same form.

The schema **Step (00)** is not provable in Linear Logic. However, there is a derivation $\mathcal{D}$ in Direct Logic of the following

Step (000)

$$\forall e\ \ell.\exists i\ j.(B(i) \wedge i \geq e) \vee (C(j) \wedge j \geq \ell), \forall x.B(x) \vee C(x),$$

$$\forall x.[[(\forall e.\exists d.B(d) \wedge d \geq e)^{(2)} \wedge B(x)] \vee \vee[(\exists b.\forall m.B(m) \supset m < b)^{(2)} \wedge C(x)]] \supset D(x)$$

$$\forall x.[(\exists b.\forall m.B(m) \supset m < b) \wedge C(x)] \supset D(x)$$

$$\forall x.[(\forall e.\exists d.B(d) \wedge d \geq e) \wedge B(x)] \supset D(x)$$

$$\vdash$$

$$(\forall k.\exists i.D(i) \wedge i \geq k)^{(3)}$$

Given a formula-occurrence A in a sequent, we use the notation

$$A^{(n)} = \underbrace{A \vee \ldots \vee A}_{n \text{ times}} \quad \text{if the occurrence of } A \text{ is negative;}$$

$$= \underbrace{A \wedge \ldots \wedge A}_{n \text{ times}} \quad \text{if the occurrence of } A \text{ is positive.}$$

Thus, $(\forall b.\exists m.B(m) \wedge m \geq b)^{(2)}$ is here $(\forall b.\exists m.B(m) \wedge m \geq b) \vee (\forall b.\exists m.B(m) \wedge m \geq b)$, etc. Moreover we use the abbreviations

df(D_1) for $[[\forall e.\exists d.B(d) \wedge d \geq e]^2 \wedge B(x)] \vee [[\exists b.\forall m.B(m) \supset m < b]^2 \wedge C(x)] \supset D_1(x)$,

df(D_2) for $[[\forall e.\exists d.B(d) \wedge d \geq e] \wedge B(x)] \supset D_1(x)$,

df(D_3) for $[[\exists b.\forall m.B(m) \supset m < b] \wedge C(x)] \supset D_3(x)$.

A proof-net corresponding to a derivation of **Step (000)** is given in Figures 3-6. The underlying informal argument is as follows.

> (*Figure* 3) Assume df(D_1), df(D_2), df(D_3) and $\forall x.B(x) \vee C(x)$, and moreover $\forall p\ q.\exists i\ j.[B(i) \wedge i \geq p] \vee [C(j) \wedge j \geq q]$. Fix an arbitrary k, we need to prove $(\exists i.D_1(i) \wedge i \geq k) \vee (\exists i.D_2(i) \wedge i \geq k) \vee (\exists i.D_3(i) \wedge i \geq k)$. Take any $r \geq k$: if $D_1(r)$, then we are done. Otherwise, both $[[\forall e.\exists d.B(d) \wedge d \geq e]^2 \wedge B(r)]$ and $[[\exists b.\forall m.B(m) \supset m < b]^2 \wedge C(r)]$ must be false and since $B(r) \vee C(r)$,

it follows that either $[\forall e.\exists d.B(d) \wedge d \geq e]^2$ or $[\exists b.\forall m.B(m) \supset m < b]^2$ is false.

(*Figure* 4) Suppose that $[\exists b.\forall m.B(m) \supset m < b]^2$ is false. We use the assumption $\mathrm{df}(D_2)$ to conclude $\exists i.D_2(i) \wedge i \geq k$. We take k for b, and let m be an arbitrary element such that $B(m) \supset m < k$ is false, thus $B(m)$ and $m \geq k$ hold. Take m for x in $\mathrm{df}(D_2)$: since the antecedent of $\mathrm{df}(D_2)\left[{x \atop m}\right]$ holds, so does the consequent $D_2(m)$, and this case is finished.

(*Figure* 5) Suppose that $[\forall e.\exists d.B(b) \wedge d \geq e]^2$ is false. We use the remaining assumptions $\mathrm{df}(D_1)$, $\forall x.B(x) \vee C(x)$ and $\forall p\ q.\exists ij.[B(i) \wedge i \geq p] \vee [C(j) \wedge j \geq q]$ to conclude $\exists i.D_3(i) \wedge i \geq k$ as follows. Let e_1 be such that $\exists d.B(b) \wedge d \geq e_1$ is false, and take e_1 for p and k for q. We obtain i and j such that $[B(j) \wedge j \geq e_1] \vee [C(i) \wedge i \geq k]$. But $[B(j) \wedge j \geq e_1]$ is impossible, therefore $C(i)$ and $i \geq k$. Take i for x in $\mathrm{df}(D_3)$: since the antecedent of $\mathrm{df}(D_3)\left[{x \atop i}\right]$ holds, so does the consequent $D_3(i)$, and the proof is finished. ■

12.2. Analysis of Step 2.

In Direc Logic we can prove

$$\boxed{\begin{array}{ll} \forall n.\mathrm{df}S(n),\ \forall x.[\forall d.d < x \wedge H_1(d) \supset S(d,x)] \supset H_1(x), & \\ \forall k.\exists i.S(k,i) \wedge i \geq k & \vdash \forall b.\exists c.H_1(c) \wedge c \geq b^2 \end{array}}$$

A proof-net corresponding to a derivation of the above sequent is given in Figure 7. The underlying informal argument is as follows.

We have a proof by Π_1-induction of

$$\forall x\ y.x < y \supset \forall z.[S(x,z) \supset S(y,z)] \tag{1}$$

using only the definition of $S(n)$. Now assume (2) $\forall x.[\forall d.d < x \wedge H_1(d) \supset S(d,x)] \supset H_1(x)$ and (3) $\forall k.\exists i.S(k,i) \wedge i \geq k$. For any b there is a c such that $S(b,c)$ and $c \geq b$, by (3). If $c \in H_1$, then we are done. Otherwise,

$\forall d.d < c \wedge H_1(d) \supset S(d,c)$ must be false, by (2). We obtain a $d < c$ such that $H_1(d)$ and not $S(d,c)$. But $S(b,c)$ and $\neg S(d,c)$ implies $d \geq b$, by (1), and the proof is finished. ∎

13. Functional Interpretation of the IRT.

In section (13.1) we give a quick estimation of the complexity of the functional interpretation. Next we construct the NCI functionals dire[illegible]ly from the formal proof described in section (12). Such constructions will yield the proof of the following theorem.

Parametrized Ramsey Theorem (PRT): *There is a functional F, primitive recursive*[9] *in χ, $\mathbf{v}$, S_F and H_1, such that for every coloring $\chi : [\mathrm{N}]^2 \to [2]$ and every choice of $\mathbf{v}$,*

(i) *if $\mathbf{v}$ satisfies the* (NTC) *and $p = F^{(l)}(\chi, \mathbf{v}, 1)$, then $[p]$ contains a χ-homogeneous set of cardinality l;*

(ii) *if $\mathbf{v}$ satisfies the* (NTC)$[x,y]$ *and $p_0, \ldots p_l \in [x,y]$, where $p_i = F^{(i)}(\chi, \mathbf{v}, p_0)$, then $[x,y]$ contains a χ-homogeneous set of cardinality l.*

Moreover, for all χ and l, there exists $\mathbf{v}$ satisfying the NTC*; in particular:*

(1) *for any fixed χ, some $\mathbf{v}$ satisfies the* NTC *for all l;*

(2) *for any fixed l, there exist p and $\mathbf{v}$ satisfying the* NTC$[0,p]$ *for all χ;*

(3) *for any fixed n_1, there exist n_2 and $\mathbf{v}$ satisfying the* NTC$[n_1,n_2]$ *for all χ, with $l = F(\chi, \mathbf{v}, n_1)$.*

Corollary 1: *The Parametrized Ramsey Theorem and Compactness*[10] *imply the Infinite Ramsey Theorem* for $c = k = 2$.

Corollary 2: *The Parametrized Ramsey Theorem implies the Finite Ramsey Theorem* for $c = k = 2$.

Corollary 3: *The Parametrized Ramsey Theorem implies the Erdős-Mills version of the Ramsey-Paris-Harrington Theorem* for $c = k = 2$.

---

[9] In fact, F is in $\mathcal{E}_2(\chi, S_F, H_1, \mathbf{v})$, the third class in the Grzegorczyk hierarchy relativized to χ, $\mathbf{v}$, S_F and H_1.

[10] Since we consider only countable colorings, when we speak of "Compactness" we mean an application of (Weak) König's Lemma.

13.1. Estimation of the complexity.

Let "*def' $S(n)$*" be the Skolemization of the definition of $S(n)$ in Step 1 and "*def'* χ^{stab}" be the Skolemization of the definition of χ^{stab} in Step 3. [11] Assume we have a formal derivation of Part 1, using *def'* $S(n)$ and *def'* χ^{stab}: *now every application of the induction rule is an instance of* Π^0_2-**IR**. The new derivation contains only one application of Π^0_2-**IR** relevant to the complexity of the functionals. Let **v** be the set of Skolem functions in question. By applying the Corollary in section 10.7, we conclude

Proposition. *The functional F such that*

$$\Gamma \vdash \forall j.H(F(j)) \wedge F(j) \geq j,$$

obtained from the interpretation of the derivation of part 1, is in $\mathcal{F}_2$, where $\mathcal{F}$ is $\{\chi, S, H_1, H, \mathbf{v}\}$, namely the third class of Grzegorczyk's hierarchy relativized to $\mathcal{F}$. ■

Remark. In Section (13.5) we show that given $\chi : [\mathrm{N}]^2 \to 2$, *by iteration of* F we can generate a χ-homogenerous set $H_*^{(l)}$ of cardinality l for certain choices of the parameters and a large χ-homogeneous set $H_*^{[n_1,n_2]}$ for other choices. We also show that $H_*^{(l)}$ and $H_*^{[n_1,n_2]}$ are bounded by the usual functions $R(2,2,l)$ and $LR(2,2,n_1)$, respectively, for any χ.

By using auxiliary colorings [12] we may obtain a proof of the IRT with exponent $k = 2$ and an arbitrary number c of colors, and then construct a functional F_c which will be in $\mathcal{F}'_{c+2}$, for an appropriate set $\mathcal{F}'$ of parameters. Then χ-homogeneous sets of given cardinality and large χ-homogeneous sets can be obtained by iteration of F_c, for arbitrary c and $\chi : [\mathrm{N}]^2 \to c$ and for appropriate choices of parameters. Now $R(c,2,l)$ is primitive recursive but $LR(c,2,c)$ is the Ackermann function. *It is clear that the computational complexity of $H_*^{(l)}$ and $H_*^{[n_1,n_2]}$ ultimately depends on the parameters.*

[11] Notice that a Skolem function for a formula becomes a Herbrand function, when the formula is written in the antecedent of a sequent.

[12] Let $\chi : [\mathrm{N}]^2 \to c$ for some c. Define auxiliary colorings χ_i for $i < c$ by letting $\chi_i(x,y) = 0$ if $\chi(x,y) = i-1$, $\chi_i(x,y) = 1$ otherwise. Then Part 1 of the proof of the IRT, with exponent $k = 2$ and $c+1$ colors, can be obtained by repeating c times Part 1 of the proof of the IRT. By the argument for the Proposition, we can construct a derivation with only c significant applications of the Π^0_2-**IR**.

13.2. Functional Interpretation, Preliminaries.

We apply the Herbrand Theorem for Direct Logic with Π_2-cut to the proof-nets considered in sections (12.1.1) and (12.2). The result provides optimal Herbrand's forms for the provable sequents **Step 1** - **Step 3** (section 12). Next, we define the functionals and produce the interpretations **Step** $\mathbf{1}_F$ - **Step** $\mathbf{3}_F$. By the correctness of the NCI we know that for any choice of numerical functions for the parameters, the functionals yield values making the *sequents* true. [13] It is reasonable to assume the following

Convention. The interpretations of the predicate letters $S(n,y)$, $H_1(x)$, $H(x)$ and of the function letter $\chi^{stab}(x)$ are sets $S_F(n)$, H_{1F}, H_F and a function f_χ^{stab} such that the sentences *def* S_F, *def* H_{1F} and *def* H_F in the sequents **Step** $\mathbf{1}_F$ - **Step** $\mathbf{3}_F$ are true. [14]

13.3. Herbrand's Analysis of Step I.

By applying Herbrand's Theorem (section 10.2) to the proof-net in figure 1 we obtain Herbrand's functions $\mathbf{p}$, $\mathbf{q}$ and a term $\mathbf{t}_0 = max(\mathbf{p},\mathbf{q})$.

Herbrand's expansion of **Step** $\mathbf{0}_0$

$$\chi(0,\mathbf{t}_0) = 0 \vee \chi(0,\mathbf{t}_0) = 1 \quad \vdash$$
$$(\chi(0,\mathbf{t}_0) = 0 \wedge \mathbf{t}_0 \geq \mathbf{p}) \vee (\chi(0,\mathbf{t}_0) = 1 \wedge \mathbf{t}_0 \geq \mathbf{q})$$

---

[13] We may assume $\chi : [\mathrm{N}]^2 \to 2$ to be true; we may restrict ourselves to terms $\chi(x,y)$ where $x < y$. These assumptions are part of the *data* of our problem: there is no point here in considering deviant interpretations of those symbols.

[14] This last condition would be guaranteed if we used the comprehension axiom in second order logic to define the sets $S(n)$, H_1 and H, instead of introducing additional predicates. Consider, for instance, the base case **Step** $\mathbf{1}_0$ in the argument by induction that proves Step 1:

$$\forall p.\chi(0,p) = 0 \vee \chi(0,p) = 1 \quad \vdash \quad \forall k.\exists i.i \in \{x : \theta(x)\} \wedge i \geq k.$$

For some θ_{jF} and some j_0, the interpretation is

$$\chi(0,t) = 0 \vee \chi(0,t) = 1 \quad \vdash \quad \vee_{j \leq j_0}(I_j \in \{x : \theta_{jF}(x)\} \wedge I_j \geq k)$$

and, by the correctness of the interpretation, $\cup_{j\leq j_0}\{x : \theta_{jF}(x)\}$ is nonempty (assuming $\chi(0,t) = 0 \vee \chi(0,t) = 1$).

Similarly, from the proof-net in figure 2 we obtain Herbrand's functions $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{a}[\mathrm{c}]$ and terms $\mathbf{t}_0 = max(\mathbf{p}, \mathbf{q})$ and $\mathbf{a}_0 = \mathbf{a}[\mathbf{t}_0]$.

Herbrand's expansion of **Step** $\mathbf{0}^n_{n+1}$

$$\chi(n+1, \mathbf{a}_0) = 0 \vee \chi(n+1, \mathbf{a}_0) = 1, S(n, \mathbf{a}_0) \wedge \mathbf{a}_0 \geq \mathbf{t}_0,$$
$$\vdash$$
$$[S(n, \mathbf{a}_0) \wedge \chi(n+1, \mathbf{a}_0) \wedge \mathbf{a}_0 \geq \mathbf{p}] \vee [S(n, \mathbf{a}_0) \wedge \chi(n+1, \mathbf{a}_0) \wedge \mathbf{a}_0 \geq \mathbf{q}].$$

Consider now figure 3. The formulas in the conclusion are not prenex, and we must use the Theorem in section (10.2.1). Therefore we regard boxes 2, 3 and 4 as resulting from boxes with prenex conclusions by substitution of suitable formulas for the atomic formulas (the conclusions of Box 1 are in prenex form). In addition, it is convenient to fix the k in the conclusion of Box 4 — we may take k to be a new constant $\mathbf{k}$ of the language $\mathcal{L}'$, coinciding with the corresponding Herbrand function. With this preparation, the following proof-structure (Box 4') yields the remaining part of Box 4 (figure 3), after indicated replacements. Here we take $r \geq k$ as an arithmetic axiom.

Box 3.

$\neg\mathrm{Df}_3, \qquad Y^2 \qquad \neg W \qquad U_3$

Box 2.

$Z^2 \qquad \neg\,\mathrm{Df}_2 \qquad U_2$

$$\dfrac{\neg B(r) \qquad \neg C(r)}{\neg B(r) \wedge \neg C(r)} \qquad \dfrac{Y^2 \qquad B(r)}{Y^2 \wedge B(r)} \qquad \dfrac{Z^2 \qquad C(r)}{Z^2 \wedge C(r)}$$

$$\neg\forall x.(B(x) \vee C(x)) \qquad \dfrac{[Y^2 \wedge B(r)] \vee [Z^2 \wedge C(r)] \qquad \dfrac{}{\neg D_1(r)}}{[Y^2 \wedge B(r)] \vee [Z^2 \wedge C(r)] \wedge \neg D_1(r)} \qquad \dfrac{D_1(r) \qquad r \geq k}{D_1(r) \wedge r \geq k}$$

$$\neg\forall x.[Y^2 \wedge B(x)] \vee [Z^2 \wedge C(x)] \supset D_1(x) \qquad \exists i.[D_1(i) \wedge i \geq k]$$

Replacements:

$[\forall e.\exists d.B(d) \wedge d \geq e]$	$\leadsto$	Y,
$[\exists b.\forall m.B(m) \supset m < b]$	$\leadsto$	Z,
$\forall x.[(\exists b.\forall m.B(m) \supset m < b) \wedge C(x) \supset D_3(x)]$	$\leadsto$	Df_3,
$\forall x.[[\forall e.\exists d.B(d) \wedge d \geq e] \wedge B(x)] \supset D_2(x)]$	$\leadsto$	Df_2,
$\forall e\ell.\exists ij.[B(i) \wedge i \geq e] \vee [C(j) \wedge j \geq \ell]$	$\leadsto$	W,
$\exists i.D_3(i) \wedge i \geq k$	$\leadsto$	U_3,
$\exists i.D_2(i) \wedge i \geq k$	$\leadsto$	U_2,

Box 4'.

Figure 9.

Similarly, Box 2 (figure 4) may be regarded as resulting from Box 2' by the same subtitutions:

$$\boxed{\text{Same as Box 1.}}$$

$$Z \qquad\qquad Y$$

$$\frac{\neg B(m_1) \qquad m_1 < k}{B(m_1) \supset m_1 < k} \qquad \frac{\dfrac{\dfrac{Y \qquad B(m_1)}{Y \wedge B(m_1)} \qquad \neg D_2(m_1)}{\neg[[Y \wedge B(m_1)] \supset D_1(m_1)]}}{\neg\forall x.[[Y \wedge B(x)] \supset D_1(x)]} \qquad \frac{\dfrac{D_2(m_1) \qquad m_1 \geq k}{D_2(m_1) \wedge m_1 \geq k}}{\exists i.D_2(i) \wedge i \geq k}$$

$$\frac{\forall m.B(m) \supset m < k}{\exists b.\forall m.B(m) \supset m < b} \qquad Z \qquad \neg\forall x.[[Y \wedge B(x)] \supset D_1(x)] \quad \exists i.D_2(i) \wedge i \geq k$$

Replacements:

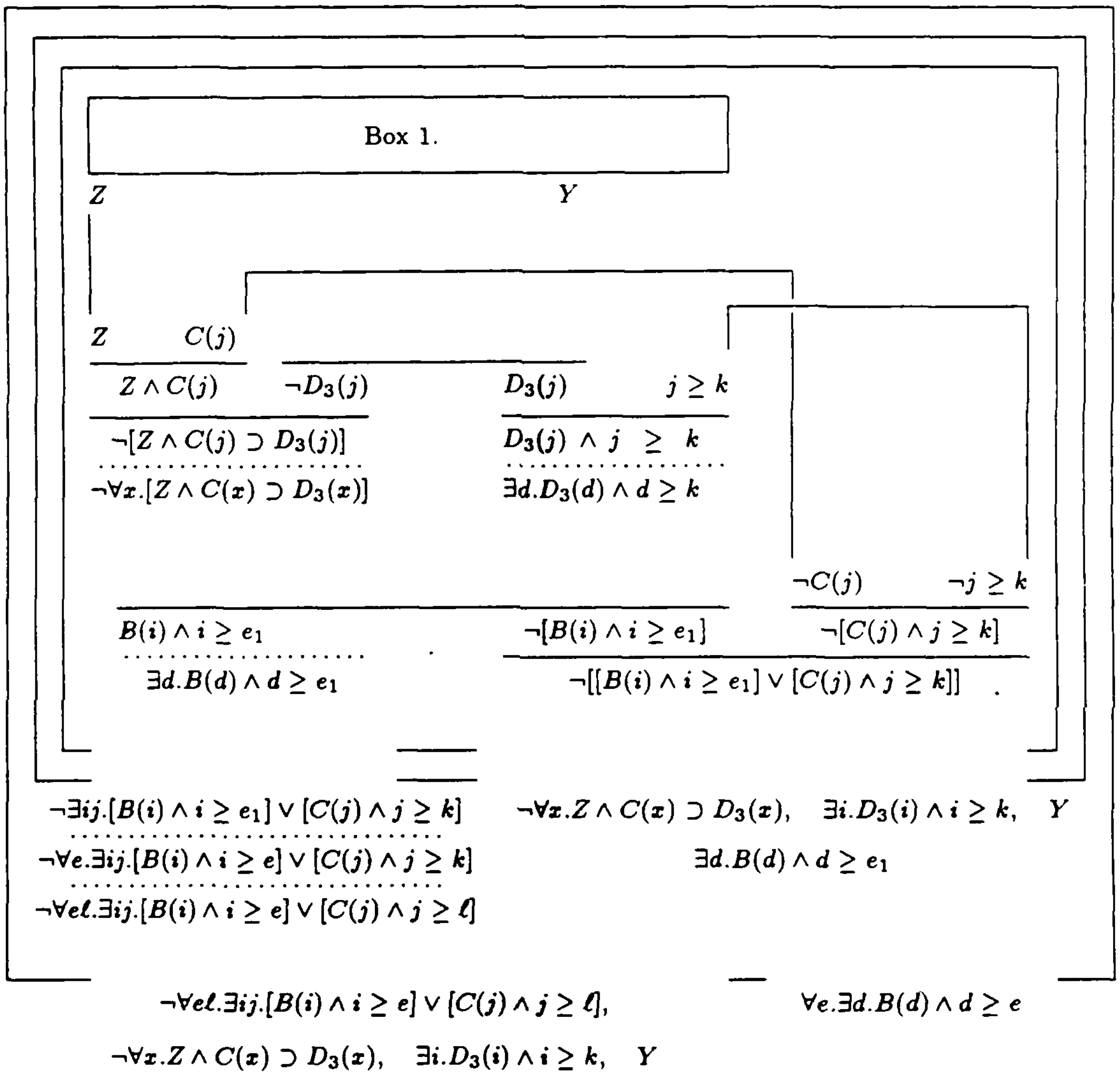
$$[\forall e.\exists d.B(d) \wedge d \geq e] \quad \rightsquigarrow \quad Y,$$

$$[\exists b.\forall m.B(m) \supset m < b] \quad \rightsquigarrow \quad Z,$$

Box 2'.

Figure 10.

A similar treatment yields Box 3 (figure 5) from Box 3'.



Box 3'.

Figure 11.

The Herbrand form of **Step (000)** is

Step $(000)_H$

$$\forall e\ \ell.(B(\mathbf{i}[e,\ell]) \wedge \mathbf{i}[e,\ell] \geq e) \vee (C(\mathbf{j}[e,\ell]) \wedge \mathbf{j}[e,\ell] \geq \ell),$$

$$\forall x.B(x) \vee C(x),$$

$$\forall x.\big[[(\exists d.B(d) \wedge d \geq \mathbf{e}[x])^{(2)} \wedge B(x)] \vee \\ [(\exists b.B(\mathbf{m}[x,b]) \supset \mathbf{m}[x,b] < b)^{(2)} \wedge C(x)]\big] \supset D_1(x)$$

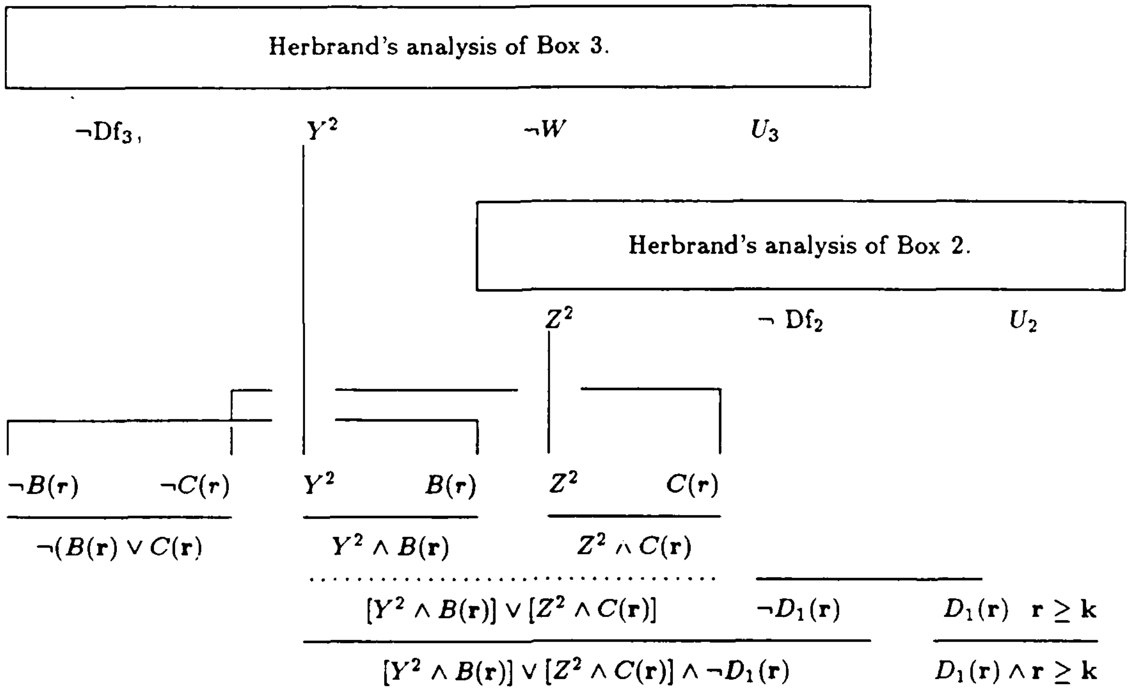
$$\forall x.[(\exists d.B(d) \wedge d \geq \mathbf{e}[x]) \wedge B(x)] \supset D_2(x)$$

$$\forall x.[(\exists b.B(\mathbf{m}[x,b]) \supset \mathbf{m}[x,b] < b) \wedge C(x)] \supset D_3(x)$$

$$\vdash$$

$$\exists i.D_1(i) \wedge i \geq \mathbf{k}, \exists i.D_2(i) \wedge i \geq \mathbf{k}, \exists i.D_3(i) \wedge i \geq \mathbf{k}$$

From the Herbrand analysis of Box 4' we obtain terms $\mathbf{k}$ and $\mathbf{r}$ and the proof-structure shown in figure 12. We assume $\mathbf{r} \geq \mathbf{k}$.



Herbrand's analysis of Box 4'.

Figure 12.

From the Herbrand analysis of Box 4' we also infer that the Herbrand terms for the eigenvariables m_1 and m_2 occurring within Box 2 are of the form $\mathbf{m}[\mathbf{r}, x]$, and that the Herbrand terms for the eigenvariables e_1 and e_3 occurring within Box 3 are of the form $\mathbf{e}[\mathbf{r}]$. In other words, we can specify the choice of Herbrand's term for **Step** $\mathbf{(000)}_H$:

$$\forall e\,\ell.(B(\mathbf{i}[e,\ell]) \wedge \mathbf{i}[e,\ell] \geq e) \vee (C(\mathbf{j}[e,\ell]) \wedge \mathbf{j}[e,\ell] \geq \ell),$$

$$B(\mathbf{r}) \vee C(\mathbf{r}),$$

$$[[(\exists d.B(d) \wedge d \geq \mathbf{e}[\mathbf{r}])^{(2)} \wedge B(\mathbf{r})] \vee$$
$$[(\exists b.B(\mathbf{m}[\mathbf{r},b]) \supset \mathbf{m}[\mathbf{r},b] < b)^{(2)} \wedge C(\mathbf{r})]] \supset D_1(\mathbf{r})$$

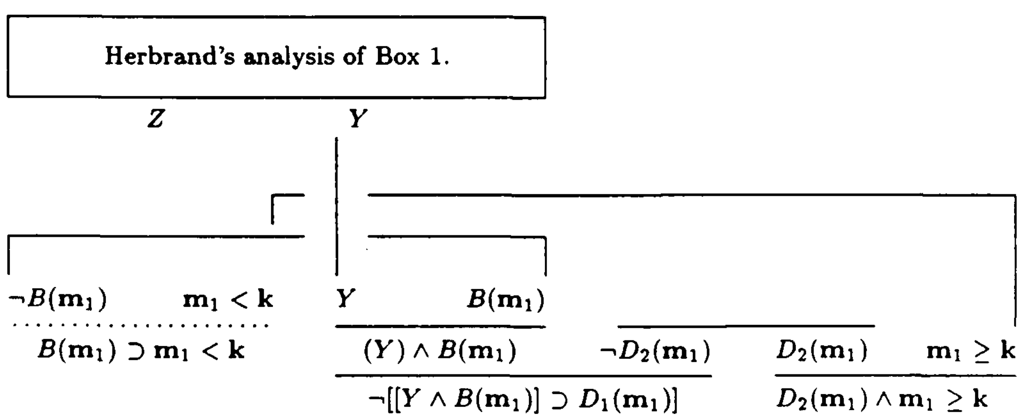
$$\forall x.[(\exists d.B(d) \wedge d \geq \mathbf{e}[x]) \wedge B(x)] \supset D_2(x)$$

$$\forall x.[(\exists b.B(\mathbf{m}[x,b]) \supset \mathbf{m}[x,b] < b) \wedge C(x)] \supset D_3(x)$$
$$\vdash$$
$$D_1(\mathbf{r}) \wedge \mathbf{r} \geq \mathbf{k}, \exists i.D_2(i) \wedge i \geq \mathbf{k}, \exists i.D_3(i) \wedge i \geq \mathbf{k}$$

From the Herbrand analysis of Box 2' we obtain terms

$$\begin{aligned} & & \mathbf{m}_1 &= \mathbf{m}[\mathbf{r},\mathbf{k}] \\ \mathbf{e}_2 &= \mathbf{e}[\mathbf{m}_1] & \mathbf{m}_2 &= \mathbf{m}[\mathbf{r},\mathbf{e}_2]. \end{aligned}$$

and the proof-structure shown in figure 13.



Herbrand's analysis of Box 2'.

Figure 13.

In figure 13 the Herbrand analysis of Box 1 is

$$\frac{B(\mathbf{m}_2) \qquad \mathbf{m}_2 \geq \mathbf{e}_2}{B(\mathbf{m}_2) \wedge \mathbf{m}_2 \geq \mathbf{e}_2} \qquad\qquad \frac{\neg B(\mathbf{m}_2) \qquad \mathbf{m}_2 < \mathbf{e}_2}{B(\mathbf{m}_2) \supset \mathbf{m}_3 < \mathbf{e}_2}$$

We obtain a further specification of the choice of Herbrand's term for **Step** $(000)_H$:

$$\begin{array}{l}
\forall e\ \ell.(B(\mathbf{i}[e,\ell]) \wedge \mathbf{i}[e,\ell] \geq e) \vee (C(\mathbf{j}[e,\ell]) \wedge \mathbf{j}[e,\ell] \geq \ell), \\[1ex]
\forall x.B(x) \vee C(x), \\[1ex]
[[(\exists d.B(d) \wedge d \geq \mathbf{e}[\mathbf{r}])^{(2)} \wedge B(\mathbf{r})] \vee \\
\quad [(B(\mathbf{m}_1) \supset \mathbf{m}_1 < \mathbf{k}) \wedge (B(\mathbf{m}_2) \supset \mathbf{m}_2 < \mathbf{e}_2) \wedge C(\mathbf{r})]] \supset D_1(\mathbf{r}) \\[1ex]
[(B(\mathbf{m}_2) \wedge \mathbf{m}_2 \geq \mathbf{e}_2) \wedge B(\mathbf{m}_1])] \supset D_2(\mathbf{m}_1) \\[1ex]
\forall x.[(\exists b.B(\mathbf{m}[x,b]) \supset \mathbf{m}[x,b] < b) \wedge C(x)] \supset D_3(x) \\
\qquad\qquad\qquad\qquad\qquad\qquad \vdash \\
D_1(\mathbf{r}) \wedge \mathbf{r} \geq \mathbf{k}, \quad D_2(\mathbf{m}_1) \wedge \mathbf{m}_1 \geq \mathbf{k}, \quad \exists i.D_3(i) \wedge i \geq \mathbf{k}
\end{array}$$

From the Herbrand analysis of Box 3' we obtain terms

$$\begin{array}{rclrcl}
\mathbf{e}_1 & = & \mathbf{e}[\mathbf{r}] & & & \\
\mathbf{i}_1 & = & \mathbf{i}[\mathbf{e}_1, \mathbf{k}] & \mathbf{j}_1 & = & \mathbf{j}[\mathbf{e}_1, \mathbf{k}] \\
\mathbf{e}_3 & = & \mathbf{e}[\mathbf{r}] & \mathbf{m}_3 & = & \mathbf{m}[\mathbf{j}_1, \mathbf{e}_3].
\end{array}$$

and the proof-structure shown in figure 14.

Herbrand's analysis of Box 1.

$Z \qquad\qquad\qquad\qquad Y$

$$\dfrac{\dfrac{Z \qquad C(\mathbf{j}_1)}{Z \wedge C(\mathbf{j}_1)} \qquad \dfrac{}{\neg D_3(\mathbf{j}_1)}}{\neg[Z \wedge C(\mathbf{j}_1) \supset D_3(\mathbf{j}_1)]} \qquad\qquad \dfrac{D_3(\mathbf{j}_1) \qquad \mathbf{j}_1 \geq \mathbf{k}}{D_3(\mathbf{j}_1) \wedge \mathbf{j}_1 \geq \mathbf{k}}$$

$$\dfrac{}{B(\mathbf{i}_1) \wedge \mathbf{i}_1 \geq \mathbf{e}_1} \qquad \dfrac{\dfrac{}{\neg[B(\mathbf{i}_1) \wedge \mathbf{i}_1 \geq \mathbf{e}_1]} \qquad \dfrac{\neg C(\mathbf{j}_1) \qquad \neg \mathbf{j}_1 \geq \mathbf{k}}{\neg[C(\mathbf{j}_1) \wedge \mathbf{j}_1 \geq \mathbf{k}]}}{\neg[[B(\mathbf{i}_1) \wedge \mathbf{i}_1 \geq \mathbf{e}_1] \vee [C(\mathbf{j}_1) \wedge \mathbf{j}_1 \geq \mathbf{k}]]}$$

Herbrand's analysis of Box 3'.

Figure 14.

In figure 14 the Herbrand analysis of Box 1 is

$$\dfrac{B(\mathbf{m}_3) \qquad \mathbf{m}_3 \geq \mathbf{e}_3}{B(\mathbf{m}_3) \wedge \mathbf{m}_3 \geq \mathbf{e}_3} \qquad\qquad \begin{array}{c} \neg B(\mathbf{m}_3) \qquad \mathbf{m}_3 < \mathbf{e}_3 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ B(\mathbf{m}_3) \supset \mathbf{m}_3 < \mathbf{e}_3 \end{array}$$

The choice of Herbrand's term for **Step** $(\mathbf{000})_H$ is now completed:

Herbrand's expansion of **Step** (000)

$$(B(\mathbf{i}_1) \wedge \mathbf{i}_1 \geq \mathbf{e}_1) \vee (C(\mathbf{j}_1) \wedge \mathbf{j}_1 \geq \mathbf{k}),$$

$$B(\mathbf{r}) \vee C(\mathbf{r}),$$

$$[[(B(\mathbf{m}_3) \wedge \mathbf{m}_3 \geq \mathbf{e}_3]) \wedge (B(\mathbf{j}_1) \wedge \mathbf{j}_1 \geq \mathbf{e}_1) \wedge B(\mathbf{r})] \vee$$
$$[(B(\mathbf{m}_2) \supset \mathbf{m}_2 < \mathbf{e}_2) \wedge (B(\mathbf{m}_1) \supset \mathbf{m}_1 < \mathbf{k}) \wedge C(\mathbf{r})]] \supset D_1(\mathbf{r})$$

$$[(B(\mathbf{m}_2) \wedge \mathbf{m}_2 \geq \mathbf{e}_2) \wedge B(\mathbf{m}_1])] \supset D_2(\mathbf{m}_1)$$

$$\forall x.[(B(\mathbf{m}_3) \supset \mathbf{m}_3 < \mathbf{e}_3) \wedge C(\mathbf{j}_1)] \supset D_3(\mathbf{j}_1)$$

$$\vdash$$

$$D_1(\mathbf{r}) \wedge \mathbf{r} \geq \mathbf{k}, \quad D_2(\mathbf{m}_1) \wedge \mathbf{m}_1 \geq \mathbf{k}, \quad D_3(\mathbf{j}_1) \wedge \mathbf{j}_1 \geq \mathbf{k}$$

13.3.1. Herbrand's Analysis of Step I, Cut.

In order to obtain the Herbrand expansion of **Step** $\mathbf{1}_0$, substitute $\mathbf{e}[\mathbf{r}]$ for $\mathbf{p}$ and $\mathbf{k}$ for $\mathbf{q}$ in the Herbrand expansion for **Step** $\mathbf{0}_0$ and in its proof-net.

Herbrand's expansion of **Step** $\mathbf{0}_0 \left[{\mathbf{p} \atop \mathbf{e}_1}\right] \left[{\mathbf{q} \atop \mathbf{k}}\right]$

$$\chi(0, \mathbf{t}) = 0 \vee \chi(0, \mathbf{t}) = 1 \quad \vdash \quad (\chi(0, \mathbf{t}) = 0 \wedge \mathbf{t} \geq \mathbf{e}_1) \vee (\chi(0, \mathbf{t}) = 1 \wedge \mathbf{t} \geq \mathbf{k})$$

where $\mathbf{e}_1 = \mathbf{e}[\mathbf{r}]$, $\mathbf{t} = max(\mathbf{e}_1, \mathbf{k})$

Replace $\chi(0, x) = 0$ for $B(x)$, $\chi(0, x) = 1$ for $C(x)$ and $S(0, x)$ for $D_1(x)$, $D_2(x)$ and $D_3(x)$ and substitute $max(x, y)$ for both $\mathbf{i}[x, y]$ and $\mathbf{j}[x, y]$ in the Herbrand expansion

of **Step (000)** and in its proof-net. We obtain terms

$$\begin{aligned} &\mathbf{k}, && \mathbf{r}, \\ \mathbf{e}_1 &= \mathbf{e}[\mathbf{r}], & \mathbf{m}_1 &= \mathbf{m}[\mathbf{r},\mathbf{k}], \\ && \mathbf{t} = max(\mathbf{e}_1, \mathbf{k}) && \\ \mathbf{e}_2 &= \mathbf{e}[\mathbf{m}_1] & \mathbf{m}_2 &= \mathbf{m}[\mathbf{r},\mathbf{e}_2]. \\ \mathbf{e}_3 &= \mathbf{e}[\mathbf{r}] & \mathbf{m}_3 &= \mathbf{m}[\mathbf{t},\mathbf{e}_3]. \end{aligned}$$

and the following expression. We may let $\mathbf{r}$ be $\mathbf{k}$.

$$(\chi(0,\mathbf{t}) = 0 \wedge \mathbf{t} \geq \mathbf{e}_1) \vee (\chi(0,\mathbf{t}) = 1 \wedge \mathbf{t} \geq \mathbf{k}),$$

$$\chi(0,\mathbf{r}) = 0 \vee \chi(0,\mathbf{r}) = 1,$$

$$\begin{aligned} &[[(\chi(0,\mathbf{m}_3) = 0 \wedge \mathbf{m}_3 \geq \mathbf{e}_3]) \wedge (\chi(0,\mathbf{t}) = 0 \wedge \mathbf{t} \geq \mathbf{e}_1) \wedge \chi(0,\mathbf{r}) = 0] \vee \\ &\quad [(\chi(0,\mathbf{m}_2) = 0 \supset \mathbf{m}_2 < \mathbf{e}_2) \wedge (\chi(0,\mathbf{m}_1) = 0 \supset \mathbf{m}_1 < \mathbf{k}) \wedge \chi(0,\mathbf{r}) = 1]] \supset \\ &\qquad\qquad \supset S(0,\mathbf{r}) \end{aligned}$$

$$[(\chi(0,\mathbf{m}_2) = 0 \wedge \mathbf{m}_2 \geq \mathbf{e}_2) \wedge \chi(0,\mathbf{m}_1]) = 0] \supset S(0,\mathbf{m}_1)$$

$$\forall x.[(\chi(0,\mathbf{m}_3) = 0 \supset \mathbf{m}_3 < \mathbf{e}_3) \wedge \chi(0,\mathbf{t}) = 1] \supset S(0,\mathbf{t})$$

$$\vdash$$

$$S(0,\mathbf{r}) \wedge \mathbf{r} \geq \mathbf{k}, \quad S(0,\mathbf{m}_1) \wedge \mathbf{m}_1 \geq \mathbf{k}, \quad S(0,\mathbf{t}) \wedge \mathbf{t} \geq \mathbf{k}$$

By applying Cut with Cut formula

$$(\chi(0,\mathbf{t}) = 0 \wedge \mathbf{t} \geq \mathbf{e}_1) \vee (\chi(0,\mathbf{t}) = 1 \wedge \mathbf{t} \geq \mathbf{k}),$$

we obtain a proof-net with conclusion

Herbrand's expansion of **Step** $\mathbf{1}_0$

$$\chi(0,\mathbf{t}) = 0 \vee \chi(0,\mathbf{t}) = 1 \quad \chi(0,\mathbf{r}) = 0 \vee \chi(0,\mathbf{r}) = 1,$$

$$\begin{aligned}&[[(\chi(0,\mathbf{m}_3) = 0 \wedge \mathbf{m}_3 \geq \mathbf{e}_3]) \wedge (\chi(0,\mathbf{t}) = 0 \wedge \mathbf{t} \geq \mathbf{e}_1) \wedge \chi(0,\mathbf{r}) = 0] \vee \\ &\quad [(\chi(0,\mathbf{m}_2) = 0 \supset \mathbf{m}_2 < \mathbf{e}_2) \wedge (\chi(0,\mathbf{m}_1) = 0 \supset \mathbf{m}_1 < \mathbf{k}) \wedge \chi(0,\mathbf{r}) = 1]] \supset \\ &\qquad\qquad \supset S(0,\mathbf{r})\end{aligned}$$

$$[(\chi(0,\mathbf{m}_2) = 0 \wedge \mathbf{m}_2 \geq \mathbf{e}_2) \wedge \chi(0,\mathbf{m}_1]) = 0] \supset S(0,\mathbf{m}_1)$$

$$\forall x.[(\chi(0,\mathbf{m}_3) = 0 \supset \mathbf{m}_3 < \mathbf{e}_3) \wedge \chi(0,\mathbf{t}) = 1] \supset S(0,\mathbf{t})$$

$$\vdash$$

$$S(0,\mathbf{r}) \wedge \mathbf{r} \geq \mathbf{k}, \quad S(0,\mathbf{m}_1) \wedge \mathbf{m}_1 \geq \mathbf{k}, \quad S(0,\mathbf{t}) \wedge \mathbf{t} \geq \mathbf{k}$$

The Herbrand analysis of **Step** $\mathbf{1}_0$ is finished.

In order to obtain the Herbrand expansion of **Step** $\mathbf{1}^n_{n+1}$, substitute $\mathbf{e}[\mathbf{r}]$ for $\mathbf{p}$ and $\mathbf{k}$ for $\mathbf{q}$ in the Herbrand expansion for **Step** $\mathbf{0}^n_{n+1}$ and in its proof-net.

Herbrand's expansion of **Step** $\mathbf{0}^n_{n+1}\begin{bmatrix}\mathbf{p}\\\mathbf{e}_1\end{bmatrix}\begin{bmatrix}\mathbf{q}\\\mathbf{k}\end{bmatrix}$

$$\chi(n+1,\mathbf{a}_1) = 0 \vee \chi(n+1,\mathbf{a}_1) = 1, S(n,\mathbf{a}_1) \wedge \mathbf{a}_1 \geq \mathbf{t},$$

$$\vdash$$

$$[S(n,\mathbf{a}_1) \wedge \chi(n+1,\mathbf{a}_1) = 0 \wedge \mathbf{a}_1 \geq \mathbf{e}_1] \vee [S(n,\mathbf{a}_1) \wedge \chi(n+1,\mathbf{a}_1) = 1 \wedge \mathbf{a}_1 \geq \mathbf{k}]$$

where $\mathbf{e}_1 = \mathbf{e}[\mathbf{r}]$, $\mathbf{t} = max(\mathbf{e}_1,\mathbf{k})$ and $\mathbf{a}_1 = \mathbf{a}[\mathbf{t}]$.

Replace $S(n,x) \wedge \chi(n+1,x) = 0$ for $B(x)$, $S(n,x) \wedge \chi(n+1,x) = 1$ for $C(x)$ and $S(n+1,x)$ for $D_1(x)$, $D_2(x)$ and $D_3(x)$ and substitute $\mathbf{a}[max(x,y)]$ for both $\mathbf{i}[x,y]$ and $\mathbf{j}[x,y]$ in the Herbrand expansion of **Step (000)** and in its proof-net. We obtain terms

$$\begin{array}{rclrcl} & \mathbf{k}, & & & \mathbf{r}, & \\ \mathbf{e}_1 & = & \mathbf{e}[\mathbf{r}], & \mathbf{m}_1 & = & \mathbf{m}[\mathbf{r},\mathbf{k}], \\ \mathbf{t} & = & max(\mathbf{e}_1,\mathbf{k}) & \mathbf{a}_1 & = & \mathbf{a}[\mathbf{t}] \\ \mathbf{e}_2 & = & \mathbf{e}[\mathbf{m}_1] & \mathbf{m}_2 & = & \mathbf{m}[\mathbf{r},\mathbf{e}_2]. \\ \mathbf{e}_3 & = & \mathbf{e}[\mathbf{r}] & \mathbf{m}_3 & = & \mathbf{m}[\mathbf{a}_1,\mathbf{e}_3]. \end{array}$$

and the following expression. We may take $\mathbf{a}_1$ for $\mathbf{r}$.

$$\begin{array}{l}
[S(n,\mathbf{a}_1) \wedge \chi(n+1,\mathbf{a}_1) = 0 \wedge \mathbf{a}_1 \geq \mathbf{e}_1] \vee \\
\vee [S(n,\mathbf{a}_1) \wedge \chi(n+1,\mathbf{a}_1) = 1 \wedge \mathbf{a}_1 \geq \mathbf{k}], \\[1ex]
S(n,\mathbf{r}) \wedge \chi(n+1,\mathbf{r}) = 0 \vee S(n,\mathbf{r}) \wedge \chi(n+1,\mathbf{r}) = 1, \\[1ex]
[[S(n,\mathbf{m}_3) \wedge \chi(n+1,\mathbf{m}_3) = 0 \wedge \mathbf{m}_3 \geq \mathbf{e}_3] \wedge \\
\quad \wedge [S(n,\mathbf{a}_1) \wedge \chi(n+1,\mathbf{a}_1) = 0 \wedge \mathbf{a}_1 \geq \mathbf{e}_1] \wedge \\
\qquad\qquad\qquad S(n,\mathbf{r}) \wedge \chi(n+1,\mathbf{r}) = 0] \vee \\[1ex]
\quad \vee [S(n,\mathbf{m}_2) \wedge \chi(n+1,\mathbf{m}_2) = 0 \supset \mathbf{m}_2 < \mathbf{e}_2] \wedge \\
\quad \wedge [S(n,\mathbf{m}_1) \wedge \chi(n+1,\mathbf{m}_1 = 0 \supset \mathbf{m}_1 < \mathbf{k}] \wedge \\
\qquad\qquad\qquad \wedge S(n,\mathbf{r}) \wedge \chi(n+1,\mathbf{r}) = 1]] \supset S(n+1,\mathbf{r}) \\[1ex]
\quad [(S(n,\mathbf{m}_2) \wedge \chi(n+1,\mathbf{m}_2) = 0 \wedge \mathbf{m}_2 \geq \mathbf{e}_2) \wedge \\
\qquad\qquad \wedge S(n,\mathbf{m}_1) \wedge \chi(n+1,\mathbf{m}_1) = 0] \supset S(n+1,\mathbf{m}_1) \\[1ex]
\forall x.[(S(n,\mathbf{m}_3) \wedge \chi(n+1,\mathbf{m}_3) = 0 \supset \mathbf{m}_3 < \mathbf{e}_3) \wedge \\
\qquad\qquad \wedge S(n,\mathbf{a}_1) \wedge \chi(n+1,\mathbf{a}_1) = 1] \supset S(n+1,\mathbf{a}_1) \\[1ex]
\qquad\qquad\qquad\qquad \vdash \\[1ex]
S(n+1,\mathbf{r}) \wedge \mathbf{r} \geq \mathbf{k}, \quad S(n+1,\mathbf{m}_1) \wedge \mathbf{m}_1 \geq \mathbf{k}, \quad S(n+1,\mathbf{a}_1) \wedge \mathbf{a}_1 \geq \mathbf{k}
\end{array}$$

By applying Cut with Cut formula

$$(\chi(0,\mathbf{t}) = 0 \land \mathbf{t} \geq \mathbf{e}_1) \lor (\chi(0,\mathbf{t}) = 1 \land \mathbf{t} \geq \mathbf{k}),$$

we obtain a proof-net with conclusion

Herbrand's expansion of **Step** $\mathbf{1}^n_{n+1}$

$$\chi(n+1,\mathbf{a}_1) = 0 \lor \chi(n+1,\mathbf{a}_1) = 1, \quad S(n,\mathbf{a}_1) \land \mathbf{a}_1 \geq \mathbf{t},$$

$$S(n,\mathbf{r}) \land \chi(n+1,\mathbf{r}) = 0 \lor S(n,\mathbf{r}) \land \chi(n+1,\mathbf{r}) = 1,$$

$$\begin{aligned}
&[[S(n,\mathbf{m}_3) \land \chi(n+1,\mathbf{m}_3) = 0 \land \mathbf{m}_3 \geq \mathbf{e}_3]\land \\
&\quad \land[S(n,\mathbf{a}_1) \land \chi(n+1,\mathbf{a}_1) = 0 \land \mathbf{a}_1 \geq \mathbf{e}_1]\land \\
&\qquad\qquad S(n,\mathbf{r}) \land \chi(n+1,\mathbf{r}) = 0]\lor \\
&\quad \lor[S(n,\mathbf{m}_2) \land \chi(n+1,\mathbf{m}_2) = 0 \supset \mathbf{m}_2 < \mathbf{e}_2]\land \\
&\quad \land[S(n,\mathbf{m}_1) \land \chi(n+1,\mathbf{m}_1 = 0 \supset \mathbf{m}_1 < \mathbf{k}]\land \\
&\qquad\qquad S(n,\mathbf{r}) \land \chi(n+1,\mathbf{r}) = 1]] \supset S(n+1,\mathbf{r})
\end{aligned}$$

$$\begin{aligned}
&[(S(n,\mathbf{m}_2) \land \chi(n+1,\mathbf{m}_2) = 0 \land \mathbf{m}_2 \geq \mathbf{e}_2)\land \\
&\qquad \land S(n,\mathbf{m}_1) \land \chi(n+1,\mathbf{m}_1) = 0] \supset S(n+1,\mathbf{m}_1)
\end{aligned}$$

$$\begin{aligned}
&\forall x.[(S(n,\mathbf{m}_3) \land \chi(n+1,\mathbf{m}_3) = 0 \supset \mathbf{m}_3 < \mathbf{e}_3)\land \\
&\qquad \land S(n,\mathbf{a}_1) \land \chi(n+1,\mathbf{a}_1) = 1] \supset S(n+1,\mathbf{a}_1)
\end{aligned}$$

$$\vdash$$

$$S(n+1,\mathbf{r}) \land \mathbf{r} \geq \mathbf{k}, \quad S(n+1,\mathbf{m}_1) \land \mathbf{m}_1 \geq \mathbf{k}, \quad S(n+1,\mathbf{a}_1) \land \mathbf{a}_1 \geq \mathbf{k}$$

The Herbrand analysis of **Step** $\mathbf{1}^n_{n+1}$ is finished.

A similar procedure yields the Herbrand expansion of **Step 3**.

13.4. Herbrand's Analysis of Step 2.

A straightforward application of Herbrand's Theorem in section (10.2) to figure 7 yields Herbrand's functions $\mathbf{d}[c]$, $\mathbf{c}[b]$ and $\mathbf{b}$, terms

$$\mathbf{c}_0 = \mathbf{c}[\mathbf{b}] \qquad \mathbf{d}_0[\mathbf{c}_0]$$

and a proof-net with the following conclusion.

Herbrand's analysis of **Step 2.**

$$[\mathbf{d}_0 < \mathbf{c}_0 \wedge H_1(\mathbf{d}_0) \supset S(\mathbf{d}_0, \mathbf{c}_0)] \supset H_1(\mathbf{c}_0),$$

$$S(\mathbf{b}, \mathbf{c}_0) \wedge \mathbf{c}_0 \geq \mathbf{b}$$

$$\vdash$$

$$H_1(\mathbf{d}_0) \wedge \mathbf{d}_0 \geq \mathbf{b}) \vee H_1(\mathbf{c}_0) \wedge \mathbf{c}_0 \geq \mathbf{b}$$

13.4.1. Functional Interpretation of Step 1.

We have exhibited in great detail the construction of proof-nets with conclusions the interpretations in Direct Logic of **Step $\mathbf{1}_0$**, **Step $\mathbf{1}^n_{n+1}$** and of **Step 2**.

Step $\mathbf{1}_0$, Direct Logic

$$\forall n\ m.\chi(n,m) = 0 \vee \chi(n,m) = 1,$$

$$\begin{aligned}\forall x.\big[[(\forall e.\exists d.\chi(0,d) = 0 \wedge d \geq e) \wedge (\forall e.\exists d.\chi(0,d) = 0 \wedge d \geq e) \wedge \chi(0,x) = 0]\vee\\ \vee[(\exists b.\forall m.\chi(0,m) = 0 \supset m < b) \wedge (\exists b.\forall m.\chi(0,m) = 0 \supset m < b)\wedge\\ \wedge\chi(0,x) = 1]\big] \supset S(0,x)\end{aligned}$$

$$\forall x.[(\exists b.\forall m.\chi(0,m) = 0 \supset m < b) \wedge \chi(0,x) = 1] \supset S(0,x)$$

$$\forall x.[(\forall e.\exists d.\chi(0,d) = 0 \wedge d \geq e) \wedge \chi(0,x) = 0] \supset S(0,x)$$

$$\vdash$$

$$\forall k.(\exists i.S(0,i) \wedge i \geq k) \vee (\exists i.S(0,i) \wedge i \geq k) \vee (\exists i.S(0,i) \wedge i \geq k)$$

Step 1^n_{n+1}, Direct Logic

$$\forall n\ q.\chi(n,q) = 0 \vee \chi(n,q) = 1, \forall n\ q.\chi(n,q) = 0 \vee \chi(n,q) = 1,$$

$$\forall c.\exists a.[S(n,a) \wedge a \geq c] \wedge [S(n,a) \wedge a \geq c]$$

$$\begin{aligned}
&\forall x.[[(\forall e.\exists d.S(n,d) \wedge \chi(n+1,d) = 0 \wedge d \geq e) \wedge \\
&\quad \wedge(\forall e.\exists d.S(n,d) \wedge \chi(n+1,d) = 0 \wedge d \geq e) \wedge \\
&\qquad\qquad \wedge S(n,x) \wedge \chi(n+1,x) = 0] \vee \\
&\quad \vee[(\exists b.\forall m.S(n,m) \wedge \chi(n+1,m) = 0 \supset m < b) \wedge \\
&\quad (\exists b.\forall m.S(n,m) \wedge \chi(n+1,m) = 0 \supset m < b) \\
&\qquad\qquad \wedge S(n,x) \wedge \chi(n+1,x) = 1]] \\
&\qquad\qquad\qquad \supset S(n+1,x)
\end{aligned}$$

$$\begin{aligned}
&\forall x.[(\exists b.\forall m.S(n,m) \wedge \chi(n+1,m) = 0 \supset m < b) \wedge \\
&\qquad\qquad \wedge S(n,x) \wedge \chi(n+1,x) = 1] \supset \\
&\qquad\qquad\qquad \supset S(n+1,x)
\end{aligned}$$

$$\begin{aligned}
&\forall x.[(\forall e.\exists d.S(n,d) \wedge \chi(n+1,d) = 0 \wedge d \geq e) \wedge \\
&\qquad\qquad \wedge S(n,x) \wedge \chi(n+1,x) = 0] \supset \\
&\qquad\qquad\qquad \supset S(n+1,x)
\end{aligned}$$

$$\vdash$$

$$\begin{aligned}
&\forall k.(\exists i.S(n+1,i) \wedge i \geq k) \vee \\
&\quad \vee(\exists i.S(n+1,i) \wedge i \geq k) \vee \\
&\quad \vee(\exists i.S(n+1,i) \wedge i \geq k)
\end{aligned}$$

Step 2, Direct Logic

$$\forall x.[\forall d.d < x \wedge H_1(d) \supset S(d,x)] \supset H_1(x)$$

$$\forall b.\exists c.S(b,c) \wedge c \geq b$$

$$\vdash$$

$$\forall b.[\exists m.H_1(m) \wedge m \geq b] \vee [\exists m.H_1(m) \wedge m \geq b]$$

We can define functionals for these sequents simply by reading off their values from the corresponding sequents in sections (13.3.1) and (13.4). For instance, for **Step $\mathbf{1}_0$** we have:

Herbrand's form of **Step $\mathbf{1}_0$, Direct Logic**

$$\forall n\ m.\chi(n,m) = 0 \vee \chi(n,m) = 1,$$

$$\begin{aligned}\forall x.\big[[(\exists d.\chi(0,d) = 0 \wedge d \geq \mathbf{e}[x])\wedge \\ \wedge(\exists d.\chi(0,d) = 0 \wedge d \geq \mathbf{e}[x]) \wedge \chi(0,x) = 0]\vee \\ \vee[(\exists b.\chi(0,\mathbf{m}[x,b]) = 0 \supset \mathbf{m}[x,b] < b)\wedge \\ \wedge(\exists b.\chi(0,\mathbf{m}[x,b]) = 0 \supset \mathbf{m}[x,b] < b) \wedge \chi(0,x) = 1]\big] \supset \\ \supset S(0,x)\end{aligned}$$

$$\forall x.[(\exists d.\chi(0,d) = 0 \wedge d \geq \mathbf{e}[x]) \wedge \chi(0,x) = 0] \supset S(0,x)$$

$$\forall x.[(\exists b.\chi(0,\mathbf{m}[x,b]) = 0 \supset \mathbf{m}[x,b] < b) \wedge \chi(0,x) = 1] \supset S(0,x) hfill$$

$$\vdash$$

$$(\exists i.S(0,i) \wedge i \geq \mathbf{k}) \vee (\exists i.S(0,i) \wedge i \geq \mathbf{k}) \vee (\exists i.S(0,i) \wedge i \geq \mathbf{k})$$

Functional form of Step 1_0, Direct Logic

$$\chi(N_0, M_0) = 0 \lor \chi(N_0, M_0) = 1,$$

$$\begin{aligned}
&[[(\chi(0, D_1) = 0 \land D_1 \geq \mathbf{e}[X_1])\land \\
&\qquad\qquad \land(\chi(0, D_2) = 0 \land D_2 \geq \mathbf{e}[X_1]) \land \chi(0, X_1) = 0]\lor \\
&\quad \lor[(\chi(0, \mathbf{m}[X_1, B_1]) = 0 \supset \mathbf{m}[X_1, B_1] < B_1)\land \\
&\qquad\qquad (\chi(0, \mathbf{m}[X_1, B_2]) = 0 \supset \mathbf{m}[X_1, B_2] < B_2) \land \chi(0, X_1) = 1]] \supset \\
&\qquad\qquad\qquad \supset S(0, X_1)
\end{aligned}$$

$$[(\chi(0, D_3) = 0 \land D_3 \geq \mathbf{e}[X_2]) \land \chi(0, X_2) = 0] \supset S(0, X_2)$$

$$[(\chi(0, \mathbf{m}[X_3, B_3]) = 0 \supset \mathbf{m}[X_3, B_3] < B_3) \land \chi(0, X_3) = 1] \supset S(0, X_3) hfill$$

$$\vdash$$

$$(S(0, I_1) \land I_1 \geq \mathbf{k}) \lor (S(0, I_2) \land I_2 \geq \mathbf{k}) \lor (S(0, I_3) \land I_3 \geq \mathbf{k})$$

where

$N_0 = 0$	$M_0 = M(\mathbf{e}, \mathbf{k}) = \mathbf{t}$
$X_1 = \mathbf{r} = \mathbf{k}$	
$D_1 = D(\mathbf{m}, \mathbf{e}, \mathbf{k}) = \mathbf{m}_3$	$D_2 = D(\mathbf{e}, \mathbf{k}) = \mathbf{t}$
$B_1 = B(\mathbf{m}, \mathbf{e}, \mathbf{k}) = \mathbf{e}_2$	$B_2 = B(\mathbf{k}) = \mathbf{k}$
$X_2 = X(\mathbf{e}, \mathbf{k}) = \mathbf{t}$	$D_3 = D(\mathbf{m}, \mathbf{e}, \mathbf{k}) = \mathbf{e}_2$
$X_3 = X(\mathbf{m}, \mathbf{k}) = \mathbf{m}_1$	$B_3 = B(\mathbf{e}, \mathbf{k}) = \mathbf{e}_3$
$I_1 = I(\mathbf{k}) = \mathbf{r}$	$I_2 = I(\mathbf{m}, \mathbf{k}) = \mathbf{m}_1$
$I_3 = I(\mathbf{e}, \mathbf{k}) = \mathbf{t}$	

We obtain functionals for **Step** $\mathbf{1}_0$ by using `if...then...else...` expressions as follows.

$$
\begin{aligned}
D_4 = D(\mathbf{m},\mathbf{e},\mathbf{k}) \quad &= \quad \texttt{if } (\chi(0,D_1)=0 \wedge D_1 \geq \mathbf{e}[X_1]),\\
&\qquad \texttt{then } D_1,\\
&\qquad \texttt{else } D_2;\\
B_4 = B(\mathbf{m},\mathbf{e},\mathbf{k}) \quad &= \quad \texttt{if } (\chi(0,\mathbf{m}[X_1,B_1])=0 \supset \mathbf{m}[X_1,B_1] < B_1),\\
&\qquad \texttt{then } B_1,\\
&\qquad \texttt{else } B_2;\\
D_0 = D(\mathbf{m},\mathbf{e},\mathbf{k}) \quad &= \quad \texttt{if } (\chi(0,D_4)=0 \wedge D_4 \geq \mathbf{e}[X_1])\vee\\
&\qquad\qquad \vee(\chi(0,\mathbf{m}[X_1,B_4])=0 \supset \mathbf{m}[X_1,B_4] < B_4),\\
&\qquad \texttt{then } D_4,\\
&\qquad \texttt{else } D_3;\\
B_0 = B(\mathbf{m},\mathbf{e},\mathbf{k}) \quad &= \quad \texttt{if } (\chi(0,D_4)=0 \wedge D_4 \geq \mathbf{e}[X_1])\vee\\
&\qquad\qquad \vee(\chi(0,\mathbf{m}[X_1,B_4])=0 \supset \mathbf{m}[X_1,B_4] < B_4),\\
&\qquad \texttt{then } B_4,\\
&\qquad \texttt{else } B_3;\\
X_0 = X(\mathbf{m},\mathbf{e},\mathbf{k}) \quad &= \quad \texttt{if } (\chi(0,D_4)=0 \wedge D_4 \geq \mathbf{e}[X_1])\vee\\
&\qquad\qquad \vee(\chi(0,\mathbf{m}[X_1,B_4])=0 \supset \mathbf{m}[X_1,B_4] < B_4),\\
&\qquad \texttt{then } X_1,\\
&\qquad \texttt{else if } [(\chi(0,D_3)=0 \wedge D_3 \geq \mathbf{e}[X_2]) \wedge \chi(0,X_2)=0],\\
&\qquad\qquad \texttt{then } X_2,\\
&\qquad\qquad \texttt{else } X_3;\\
I_0 = I(\mathbf{m},\mathbf{e},\mathbf{k}) \quad &= \quad \texttt{if } (S(0,I_1) \wedge I_1 \geq \mathbf{k}),\\
&\qquad \texttt{then } I_1,\\
&\qquad \texttt{else if } (S(0,I_2) \wedge I_2 \geq \mathbf{k}),\\
&\qquad\qquad \texttt{then } I_2,\\
&\qquad\qquad \texttt{else } I_3.
\end{aligned}
$$

We proceed similarly in the induction step, and we define the functional I_{ind} as

$$I_{ind}(\mathbf{m},\mathbf{e},\mathbf{k},n) = \begin{cases} r, & \text{if } S_F(n+1,r) \wedge r \geq k; \\ \mathbf{m}_1, & \text{if } S_F(n+1,\mathbf{m}_1) \wedge \mathbf{m}_1 \geq \mathbf{k}; \\ \mathbf{a}_1 & \text{otherwise.} \end{cases}$$

Here $\mathbf{a}_1$ is $\mathbf{a}[n,t]$; $\mathbf{a}[n,x]$ is a parameter deriving from the interpretation of the inductive hypothesis $\forall c.\exists a.S(n,a) \wedge a \geq c$ – namely, $S(n,\mathbf{a}_1) \wedge \mathbf{a}_1 \geq t$ –. Finally, we define the functionals B, D, X and I corresponding to the conclusion of the Induction Rule, **Step 1**. For instance, define I to be

$$\begin{aligned} I(0,\mathbf{m},\mathbf{e},\mathbf{k}) =& I_0(\mathbf{m},\mathbf{e},\mathbf{k}) \\ I(n+1,\mathbf{m},\mathbf{e},\mathbf{k}) =& I_{ind}(\lambda k.I(n,\mathbf{m},\mathbf{e},k),\mathbf{m},\mathbf{e},\mathbf{k},n) \end{aligned}$$

by primitive recursion. This concludes the interpretation of Step 1. The interpretation of Steps 2 and 3 is omitted.

13.5. Choices for the "Hidden Parameters".

Before giving a formal characterization of the Non-Triviality Condition, we consider some examples of evaluation of the functionals. Here let $H_* = \{F(1), F^2(1), ...\} \subset H_F$ be the result of iterating the functional $\lambda n.F(\chi,\mathbf{m},\mathbf{e},\mathbf{q},\mathbf{u},n)$ extracted from the proof of the Infinite Ramsey Theorem, for the given values of the parameters **m**,**e**,**q**,**u**, n; write $H_*^{[x,y]}$ for $H_* \cap [x,y]$ and let $H_*^{(n)}$ be the χ-homogeneous set given by n iterations of F.

Example 0. Let $m[n,x,y] = max(n+1,y)$, $e[n,x] = 0$. Then in the Herbrand expansion of **Step $\mathbf{1}_0$** we have $e_1 = e_2 = e_3 = 0$, $m_1 = t = k$, $m_2 = m_3 = 1$ and the defining conditions for S_F are tautological, thus S_F is $\mathbf{N}$. Similarly for the expansion of **Step $\mathbf{1}^n_{n+1}$**. Thus $S_F(n) = \mathbf{N}$ for all n, and so $H_{1\,F} = \mathbf{N}$, too. Not surprisingly, the iteration of F in this case grows linearly.

Example 1: As usual, let $S(-1) = \mathbf{N}$, *Green* $(n) = \{z : \chi(n,z) = 0\}$ and *Red* $(n) = \{z : \chi(n,z) = 1\}$ (by our assumption, for all z in *Green* (n) or *Red* (n), $z > n$). Let

$$e[n,x] = \begin{cases} 0, & \text{if } Green\ (n) \cap S(n-1) \text{ is unbounded;} \\ max(Green\ (n) \cap S(n-1)) & \text{otherwise.} \end{cases}$$

$$m[n,x,y] = \begin{cases} \mu z.z \in Green\ (n) \cap S(n-1) \wedge z > y & \\ & \text{if } Green\ (n) \cap S(n-1) \text{ is unbounded;} \\ 0 & \text{otherwise.} \end{cases}$$

Functions m and e satisfying the above conditions are in fact Skolem functions for *def* $S(n)$, the defining conditions of $S(n)$, as well as for *def* χ^{stab} and so "preserve the meaning" of those definitions. There is nothing to verify here.

Notice that for this m and e we can choose colorings that make $F^{(l)}$ – the l-th iteration of F – grow as fast as desired, i.e., this choice gives no bound for $R(2,2,l)$. Also notice that $H_*^{(n+1)}$ is an extension of $H_*^{(n)}$; thus from the set H_* by compactness we obtain again the Infinite Ramsey Theorem, as expected.

Example 2: Let $S(-1) = [p]$, *Green* (n) and *Red* (n) as before.

$$e[n,x] = \begin{cases} 0, & \text{if } |Green\ (n) \cap S(n-1)| \geq |Red\ (n) \cap S(n-1)|; \\ p & \text{otherwise.} \end{cases}$$

$$m[n,x,y] = \begin{cases} \mu z < p.z \in Green\ (n) \cap S(n-1) \wedge z > y \\ \qquad \text{if } |Green\ (n) \cap S(n-1)| \geq |Red\ (n) \cap S(n-1)|; \\ \mu z < p.z \in Red(n) \cap S(n-1), \\ \qquad \text{otherwise.} \end{cases}$$

If $p \geq 2^{2l-1}$, the above choice for $\mathbf{m}$ and $\mathbf{e}$ makes $S_F(n) \cap [p]$ satisfy the definition of $S(n)$ in the Finite Ramsey Theorem and *as k varies over* $[p]$, $I(n,m,e,k)$ generates the elements of $S(n)$ — notice that for $y \geq p$, $m[n,x,y] = 0$: considering the Herbrand expansion of **Step** $\mathbf{1}_0$ we see that (1) becomes a tautology for $k \geq p$; thus $\mathbf{N} \setminus [p-1] \subset S_H(0)$ —. We may conclude that with this choice of parameters (and with suitable choices for $\mathbf{q}$ and $\mathbf{u}$) by iterating the functional F l times we generate a χ-homogeneous set $H_*^{(l)}$. Following the proof of the Finite Ramsey Theorem (section 11.2), we can check that $H_*^{(l)} \subset [p]$ for any χ, i.e., the familiar bound is preserved.

Example 3: Let $[n_1, n_2]$, $a_0, \ldots, a_p$, $b_0, \ldots, b_q$, $A_1, \ldots, A_p$, $B_1, \ldots, B_q$ be as in section (11.3) and consider the definition by cases of $S(n)$ given in Proof 2, section (11.3).

$$e[n,x] = \begin{cases} 0 & \text{if } S(n) = Green(n) \cap S(n-1); \\ n_2 & \text{otherwise.} \end{cases}$$

$$m[n,x,y] = \begin{cases} \mu z < n_2.z \in Green\ (n) \cap S(n-1) \wedge z > y \\ \qquad \text{if } S(n) = Green(n) \cap S(n-1); \\ \mu z < n_2.z \in Red\ (n) \cap S(n-1) \\ \qquad \text{otherwise.} \end{cases}$$

As before, we check that with this choice of parameters (and with suitable choices for $\mathbf{q}$ and $\mathbf{u}$) $S_F(n) \cap [n_1, n_2]$ satisfies the definition of $S(n)$ in the RPH theorem and

as k varies over $[n_1, n_2]$ $I(n, m, e, k)$ generates the elements of $S(n)$ and F generates a χ-homogeneous large set $H_*^{[x,y]}$ for any χ, i.e., the familiar bound is preserved.

From the examples we see that our computation will produce one of the following outcomes, depending on the choice of the parameters and $\mathbf{m}$, $\mathbf{e}$:

(i) as k varies, the functional $I(n, m, e, k)$ gives values both in *Green* (n) and in *Red* (n) and the set H_* is not χ-homogeneous;

(ii) as k varies, the functional $I(n, m, e, k)$ gives values only in *Green* (n) or in *Red* (n), and thus the set H_* is χ-homogeneous;

(iii) as k varies within a segment $[x, y]$ of $\mathbf{N}$, the functional $I(n, m, e, k)$ gives values only in *Green* (n) or in *Red* (n), thus the set H_* is χ-homogeneous in the segment $[x, y]$.

Finally, information about the cardinality of a certain segment of H_0 requires additional proof. In example 2 we needed to know that $p = r(l)$ in order to argue that $|H_*^{[0,p]}| \geq l$ independenly of χ and in example 3 we used the fact that $n_2 = lr(n_1)$ in arguing that $H_*^{[n_1,n_2]}$ is large, for any χ.

The condition for the "non-triviality" of the computation is easily spelled out. Given a certain choice of $\mathbf{m}$ and $\mathbf{e}$, let

$$\begin{aligned}
\rho(0,k) =& [(\chi(0, \mathbf{m}[0, X, B]) = 0 \wedge \mathbf{m}[0, X, B] \geq B) \wedge \chi(0, X) = 0] \\
\rho(n+1,k) =& [(S(N, \mathbf{m}[n+1, X, B]) \wedge \chi(n+1, \mathbf{m}[n+1, X, B]) = 0 \wedge \\
& \wedge \mathbf{m}[n+1, X, B] \geq B) \wedge S(n, X) \wedge \chi(n+1, X) = 0] \\
\sigma(0,k) =& [(\chi(0, D) = 0 \supset D < \mathbf{e}[0, X]) \wedge \chi(0, X) = 1] \\
\sigma(n+1,k) =& [(S(n, D) \wedge \chi(n+1, D) = 0 \supset D < \mathbf{e}[n+1, X]) \wedge \\
& \wedge S(n, X) \wedge \chi(n+1, X) = 1]
\end{aligned}$$

(here B, D and X depend on n and k, and by definition of X the value $X = k$ is tested). For the "non-triviality" of F (over a certain segment $[x, y]$) we need

(NTC): for all n, if $S(n-1, n)$, then $(\forall k.\rho(n, k)) \vee (\forall k.\sigma(n, k)$ and $\forall k.\rho(n, k) \equiv \neg\sigma(n, k)$.

(NTC)$[x, y]$: for all $n \in [x, y]$, if $S(n-1, n)$, then $(\forall k_{x \leq k \leq y}.\rho(n, k)) \vee (\forall k_{x \leq k \leq y}.\sigma(n, k)$ and $\forall k_{x \leq k \leq y}.\rho(n, k) \equiv \neg\sigma(n, k)$.

The Parametrized Ramsey Theorem (section 1.2.1) is now proved. ■

14. The Compactness Argument.

In the last section we have made choices of parameters **m**, **e**, **q** and **u**, relying on well-known facts of finite combinatorics and we have shown that for those choices the desired sets are generated by iteration of the functionals within known bounds. It is conceivable that if we work with a compactness argument, there may be implicit in the part of the proof to which the IRT is applied (1) a particular choice of a coloring χ_0 and (2) specific numeric functions for **m**, **e**, **q** and **u**, such that they may be mechanically uncovered by some proof-theoretic transformation. We might hope that (1) the coloring χ_0 would give information about a "worst case coloring" or (2) the numeric functions for **m**, **e**, **q** and **u** would be choice functions operating on an initial fragment of χ_0. In these cases the answer to *Question 1* (section 11.4) would be positive.[15] Part of the problem is to decide what the compactness argument is - different mathematical techniques may produce different proofs and different bounds. An obvious choice is the use of König's Lemma.

Compactness Argument. Assume the negation of the FRT, namely for given c, k, l,

$\neg\mathrm{FRT}^{(k)}$: *for each n there is a coloring $\xi_n : [n]^k \to c$ such that every set $H \subset [n]$ of cardinality l is not ξ_n-homogeneous.*

The set of finite colorings ξ_n falsifying the FRT, ordered by the relation "*ξ_i is extended by ξ_j*", forms a tree, which by our assumption is infinite. At each node $\xi_n : [n]^k \to c$ there are at most $c^{\binom{n}{k-1}}$ extensions $\xi' : [n+1]^k \to c$ of ξ_n. By König's Lemma there is an infinite path through our tree, i.e., a coloring $\chi : [\mathrm{N}]^k \to c$ such that every set H of cardinality l is not χ-homogeneous. This contradicts $\mathrm{IRT}^{(k)}$, the IRT with exponent k. ■

The above argument may be formalized by a derivation $\mathcal{D}$ in $\mathbf{PA}^2$, the sequent calculus for second order **PA** as follows. Let KL be a formalization of König's

15 Although a detailed analysis is beyond the scope of this paper, the following remarks provide evidence for our belief that the results presented in the previous sections are optimal for the techniques under consideration.

Lemma. [16] Let $\mathcal{D}_1^k$ and $\mathcal{D}_2$ be derivations in $\mathbf{PA}^2$ of $\mathrm{IRT}^{(k)}$ and KL, respectively; let $\mathcal{D}_0$ be a derivation of

$$(\diamond) \qquad \mathrm{IRT}^{(k)}, \mathrm{KL}, \neg\mathrm{FRT}^{(k)} \vdash .$$

in $\mathbf{LK}^2$, the sequent calculus for pure second order logic. Now $\mathcal{D}$ results from $\mathcal{D}_1^k$, $\mathcal{D}_2$ and $\mathcal{D}_0$ by two applications of Cut. We are interested in the following transformation: apply Cut-Elimination to $\mathcal{D}$ and then the No Counterexample Interpretation to the resulting derivation $\mathcal{D}'$.

Notice that $\mathrm{FRT}^{(k)}$ has the form $\exists n.\forall\xi_n.\phi(\xi_n, n)$ and that the NCI of $\mathcal{D}'$, yields a functional N such that $\phi_F(\xi[N], N)$ is true for all choice of numerical functions for the parameters. Here ξ is a new $k+1$-ary function parameter, representing an infinite sequence of finite colorings $\{\xi_n : [n]^k \to c\}_n$, i.e., any attempted counterexample to the FRT. Moreover, ξ may be regarded as an infinite coloring $\xi : [\mathrm{N}]^{k+1} \to c$ by

$$(*) \qquad \xi(i_0, \ldots, i_k) = \xi_{i_k}(i_0, \ldots, i_{k-1})$$

(we assume $i_0 < \ldots < i_k$).

We have not checked this in detail, but we conjecture a negative answer to *Question 1* on the basis of the following experiment. The Finite Ramsey Theorem with exponent k follows from the Infinite Ramsey Theorem with exponent $k+1$. Let $\mathcal{D}_1^{k+1}$ be a derivation of $\mathrm{IRT}^{(k+1)}$ and let $\mathcal{D}_*$ be a derivation in $\mathbf{LK}^2$ of

$$(\triangle) \qquad \mathrm{IRT}^{(k+1)} \vdash \forall\zeta.\exists n.\phi(\zeta(n), n).$$

If $\mathcal{D}$ is the derivation ending with an application of Cut to the conclusions of $\mathcal{D}_1^{k+1}$ and $\mathcal{D}_*$, then it is easy to see that after the elimination of this Cut and application of the NCI, the functional N still depends on the coloring ζ — which is arbitrary — and on the *parameters* **m** and **e** — rather than on specific numeric functions. [17]

[16] We write $\mathrm{IRT}^{(k)}$ and $\mathrm{FRT}^{(k)}$ for their formalization, too.

[17] The significance of this experiment for an analysis of the compactness argument follows from the following. (1) *Ceteris paribus*, the NCI of a derivation of $\vdash \exists n.\forall\xi_n.\phi(\xi_n, n)$ has the same form as the the NCI of a derivation of $\vdash \forall\zeta.\exists n.\phi(\zeta(n), n)$. (2) Following Kunen [1977], the argument in the proof of the Infinite Ramsey Theorem for the existence of an infinite pre-homogeneous set for $\xi : [\mathrm{N}]^{k+1} \to c$ is in fact an application of König's Lemma. In our terminology and in the case of $k = 2$, this is the argument that establishes the existence of the infinite set H_1 - corresponding to Steps 1 and 2.

15. Conclusion.

What conclusions may we draw from our experiment? Our application of Proof Theory to Combinatorics has produced the Parametrized Ramsey Theorem, a general logical frame for "Ramsey-type" results. Additional computational instructions, in the form of choice-function parameters, yield the Infinite, the Finite and the Paris-Harrington versions of Ramsey's Theorem (see *Question 2*).

From this exercise we have not obtained new bounds for the Finite Ramsey Theorem. [18] It is an open question whether one can provide *new* function parameters for which the interpretation is non-trivial on a certain segment $[x, y]$ and such that $\{F(x), \ldots, F^{(l)}(x)\} \subset [x, y]$ for any coloring χ. [19]

We have considered fragments of arithmetics in which the No Counterexample Interpretation has a simple definition and the functionals are primitive recursive in the function parameters. In particular, using results by Sieg we have obtained a straigtforward estimation of the complexity of the functional interpretation of the Infinite Ramsey Theorem (section 13.1).

We have shown that Ketonen's Direct Logic and Girard's Linear Logic can be used to keep cumbersome computations under control – in these logics Herbrand's Theorem has minimal complexity and for proofs in Π^0_2-**IR** the result of evaluating the functionals is invariant under elimination of Π^0_2-Cuts.

The interest of this approach is increased by the fact that Direct Logic is the logic of the decision procedure of the proof-checker *EKL* (see Ketonen [1983]); thus implementation in *EKL* of the functional interpretation will make extensive experimentation more accessible.

[18] The functional obtained from the Infinite Ramsey Theorem depends on the given coloring $\chi : [\mathrm{N}]^2 \to 2$. By choosing suitable function parameters we made the functional depend only on an initial segment of χ, but the bound was explicitly provided with the parameters rather than extracted from the proof. On the other hand, the application of the functional interpretation to the compactness argument from the Infinite to the Finite Ramsey Theorem introduces a new parameter ξ that may be regarded as an arbitrary coloring $\chi : [\mathrm{N}]^3 \to 2$. We have been unable to mechanically extract choice-functions depending on an initial segment of ξ only.

[19] On the other hand, significant improvement on these bounds are obtained using a completely different proof idea for the Finite Theorem that has no analogue in the Infinite Theorem (see the alternative proof by double induction in Graham et al. [1980], p.3).

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